MINISTRY OF EDUCATION

Additional Mathematics

TEACHER MANUAL

YEAR 1 - BOOK 1

NATIONAL COUNCIL FOR CURRICULUM & ASSESSMENT OF MINISTRY OF EDUCATION

MINISTRY OF EDUCATION

REPUBLIC OF GHANA

Additional Mathematics

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NATIONAL COUNCIL FOR
CURRICULUM & ASSESSMENT
OF MINISTRY OF EDUCATION

ADDITIONAL MATHEMATICS TEACHER MANUAL

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INTRODUCTION

The National Council for Curriculum and Assessment (NaCCA) has developed a new Senior High School (SHS), Senior High Technical School (SHTS) and Science, Technology, Engineering and Mathematics (STEM) Curriculum. It aims to ensure that all learners achieve their potential by equipping them with 21st Century skills, competencies, character qualities and shared Ghanaian values. This will prepare learners to live a responsible adult life, further their education and enter the world of work.

This is the first time that Ghana has developed an SHS Curriculum which focuses on national values, attempting to educate a generation of Ghanaian youth who are proud of our country and can contribute effectively to its development.

This Teacher Manual for Additional Mathematics covers all aspects of the content, pedagogy, teaching and learning resources and assessment required to effectively teach Year One of the new curriculum. It contains this information for the first 12 weeks of Year One, with the remaining 12 weeks contained within Book Two. Teachers are therefore to use this Teacher Manual to develop their weekly Learning Plans as required by Ghana Education Service.

Some of the key features of the new curriculum are set out below.

Learner-Centred Curriculum

The SHS, SHTS, and STEM curriculum places the learner at the center of teaching and learning by building on their existing life experiences, knowledge and understanding. Learners are actively involved in the knowledge-creation process, with the teacher acting as a facilitator. This involves using interactive and practical teaching and learning methods, as well as the learner's environment to make learning exciting and relatable. As an example, the new curriculum focuses on Ghanaian culture, Ghanaian history, and Ghanaian geography so that learners first understand their home and surroundings before extending their knowledge globally.

Promoting Ghanaian Values

Shared Ghanaian values have been integrated into the curriculum to ensure that all young people understand what it means to be a responsible Ghanaian citizen. These values include truth, integrity, diversity, equity, self-directed learning, self-confidence, adaptability and resourcefulness, leadership and responsible citizenship.

Integrating 21st Century Skills and Competencies

The SHS, SHTS, and STEM curriculum integrates 21st Century skills and competencies. These are:

- **Foundational Knowledge:** Literacy, Numeracy, Scientific Literacy, Information Communication and Digital Literacy, Financial Literacy and Entrepreneurship, Cultural Identity, Civic Literacy and Global Citizenship
- **Competencies:** Critical Thinking and Problem Solving, Innovation and Creativity, Collaboration and Communication
- **Character Qualities:** Discipline and Integrity, Self-Directed Learning, Self-Confidence, Adaptability and Resourcefulness, Leadership and Responsible Citizenship

Balanced Approach to Assessment - not just Final External Examinations

The SHS, SHTS, and STEM curriculum promotes a balanced approach to assessment. It encourages varied and differentiated assessments such as project work, practical demonstration, performance assessment, skills-based assessment, class exercises, portfolios as well as end-of-term examinations and final external assessment examinations. Two levels of assessment are used. These are:

- o Internal Assessment (30%) – Comprises formative (portfolios, performance and project work) and summative (end-of-term examinations) which will be recorded in a school-based transcript.
- o External Assessment (70%) – Comprehensive summative assessment will be conducted by the West African Examinations Council (WAEC) through the WASSCE. The questions posed by WAEC will test critical thinking, communication and problem solving as well as knowledge, understanding and factual recall.

The split of external and internal assessment will remain at 70/30 as is currently the case. However, there will be far greater transparency and quality assurance of the 30% of marks which are school-based. This will be achieved through the introduction of a school-based transcript, setting out all marks which learners achieve from SHS 1 to SHS 3. This transcript will be presented to universities alongside the WASSCE certificate for tertiary admissions.

An Inclusive and Responsive Curriculum

The SHS, SHTS, and STEM curriculum ensures no learner is left behind, and this is achieved through the following:

- Addressing the needs of all learners, including those requiring additional support or with special needs. The SHS, SHTS, and STEM curriculum includes learners with disabilities by adapting teaching and learning materials into accessible formats through technology and other measures to meet the needs of learners with disabilities.
- Incorporating strategies and measures, such as differentiation and adaptative pedagogies ensuring equitable access to resources and opportunities for all learners.
- Challenging traditional gender, cultural, or social stereotypes and encouraging all learners to achieve their true potential.
- Making provision for the needs of gifted and talented learners in schools.

Social and Emotional Learning

Social and emotional learning skills have also been integrated into the curriculum to help learners to develop and acquire skills, attitudes, and knowledge essential for understanding and managing their emotions, building healthy relationships and making responsible decisions.

Philosophy and vision for each subject

Each subject now has its own philosophy and vision, which sets out why the subject is being taught and how it will contribute to national development. The Philosophy and Vision for Additional Mathematics is:

Philosophy: Learners can develop their potential in Additional Mathematics through creative, innovative and interactive ways to become lifelong learners, apply mathematical skills and competencies to solve everyday problems, further their education to read mathematics-related courses and/or proceed to the world of work.

Vision: Learners enthusiastic about mathematics, capable of reasoning (quantitatively and abstractly), modelling, representing and using mathematical skills, tools and technology to solve real life problems, further their studies and/or proceed to the world of work.

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SCOPE AND SEQUENCE

Additional Mathematics Summary

S/N	STRAND	SUB-STRAND	YEAR 1			YEAR 2			YEAR 3		
			CS	LO	LI	CS	LO	LI	CS	LO	LI
1	Modelling with Algebra	Number and Algebraic Patterns	2	4	13	2	7	8	-	-	-
		Applications of Algebra	2	8	21	2	6	19	2	2	8
2	Geometric Reasoning and Measurement	Spatial Reasoning	2	4	11	2	2	8	1	4	11
		Measurement of Triangles	1	2	6	1	2	4	2	1	4
3	Calculus	Principles of Calculus	1	1	6	2	2	9	1	1	4
		Applications of Calculus	1	1	2	1	1	2	1	4	4
4	Handling data	Organising, Representing and Interpreting Data	1	2	8	2	2	4	1	3	9
		Making Predictions with Data	1	2	7	2	2	6	1	2	6
Total			**11**	**24**	**73**	**14**	**22**	**60**	**9**	**14**	**46**

Overall Totals (SHS 1 – 3)

Content Standards	**33**
Learning Outcomes	**61**
Learning Indicators	**180**

SECTION 1: BINARY OPERATIONS, SETS AND BINOMINAL EXPRESSIONS

Strand: **Modelling with Algebra**

Sub-Strand: Number and Algebraic Patterns

Content Standard: Demonstrate knowledge and understanding of numbers involving the use of binary operations, sets and binomial theorem to solve problems in real life situations

Learning Outcome(s):

a. *Solve problems involving properties of binary operations*

b. *Model and solve real life problems on sets*

c. *Expand binomials with positive integral indices and simplify coefficients of the terms*

INTRODUCTION AND SECTION SUMMARY

In this section, we will discuss binary operations, sets and binomial expressions. In binary operation, we will discuss the concepts and properties such as closure, associativity, commutativity, distributivity, identity elements and inverse. Binary operations are used extensively in algebra, trigonometry, and calculus, forming the backbone of numerous calculations. Sets are essential for organizing and classifying information and are used in areas like probability and statistics. Our focus will be on operations on three set problems and set algebra. The Binomial theorem provides a powerful formula to expand expressions of the form $(a + b)^n$, where n is a non-negative integer. It allows us to calculate the coefficients and exponents of the expanded expression. Binomial expansions are crucial for solving various problems in algebra and beyond. The concepts in this section are essential for science, algebra, trigonometry, and calculus, forming the backbone of numerous calculations in finance to model compound interest, etc.

***The weeks covered by the section are**:*

Week 1:

a. *The concept of binary operations*

b. *Properties of binary operations*

c. *Identity and the inverse of an element*

d. *Real-life problems involving binary operation*

Week 2:

a. *Sets, properties and operations on sets*

b. *Venn diagrams involving three sets*

Week 3:

a. *Defining and expanding binomial expressions*

b. *Application of binomial expansions*

SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in practical activities on binary operations, sets and binomial expressions. Learners should be given the platform to work in groups to develop their own real-life questions and find answers. Therefore, experiential learning activities and mixed-ability groupings should dominate the lessons on these concepts. All learners, irrespective of their learning abilities should be assisted to take part fully in investigations and presentation of findings. However, make considerations and accommodations for the different groups. That is, offer approaching proficiency learners the opportunity to make oral presentations. Then, extend activities for the above average/highly proficient learners to use formulae and computer applications to solve problems.

ASSESSMENT SUMMARY

Assessment methods, ranging from quizzes, tests, and homework assignments, can be used to evaluate learners understanding of concepts and their ability to solve problems. Performance tasks like solving real-world problems involving binary operations, sets and binomial expansions will also be used to assess learner's application of these mathematical skills. Also sets (properties, operations and Venn diagrams) will make use of various visual aids such as diagrams of types of sets; and charts on Venn diagrams and set properties and interactive tools on types of sets and their operations will also be incorporated to engage learners in hands-on learning experiences. Assessment strategies which vary from Level 1 to Level 4 questions of the DoK will be used. Teachers should record the performances of learners for continuous assessment records.

Week 1

Learning Indicator(s):

a. *Recognise binary operations, and apply the knowledge in solving related problems*
b. *Describe and interpret the characteristics of commutative, associative, distributive and closure properties of binary operations.*
c. *Determine the identity element and use it to find the inverse of a given element*

Theme or Focal Area: The concept of binary operations

The study of binary operations dates back to ancient times, where various civilizations explored numerical systems and mathematical operations. However, the concept of binary operations as we understand them today emerged in the 19th and 20th centuries with the development of formal algebraic structures. A binary operation is a rule which combines elements from a non-empty set ℝ to produce another element. Binary operations form the foundation of digital logic circuits. The knowledge of binary operations is essential in computer science and programming and other fields of science. Learning binary operations fosters computational thinking skills, including logical reasoning, pattern recognition, and algorithmic problem-solving.

Binary operations can be defined using symbols such as $\otimes$, $\boxplus$, $\circledast$, $*$ and $\diamond$. For example, "An operation is such that the result is the difference of the product of two numbers and the sum of the two numbers". The statement can be written as a mathematical rule thus:

$a * b = ab - (a + b)$ where a and b are the two numbers and $*$ is the symbol that represents the operation

Example
Given that $m * n = \frac{m+n}{m}$, evaluate $(2 * 3)$

Solution:
$m * n = \frac{m+n}{m}$

$2 * 3 = \frac{2+3}{2} = \frac{5}{2}$

Theme or Focal Area: Properties of binary operations

The properties of binary operations which will be discussed under this theme are closure, commutativity, associativity and distributive property. We will investigate these properties of the four basic operations $(\times , \div, -, +)$ on the set of real numbers $(\mathbb{Q}, \mathbb{Z}, \mathbb{W}, \mathbb{N})$ and predefined binary operation on a given set using algebraic manipulations and binary tables.

Verifying properties of binary operations

a. Closure

A binary operation, Δ is closed under a set, A if for all x and y which are elements of A, $x \Delta y$ is also in A. For example, the sum of two natural numbers is a natural number hence it can be said that addition is closed under the set of natural numbers, $\mathbb{N}$

b. Commutativity

Given that $\diamond$ is a binary operation defined on a set, S which contains a and b, if $a \diamond b = b \diamond a$, for all a and b in S then $\diamond$ is said to be commutative. In a binary table, the binary operation is only commutative if the results of the operation are symmetrical about the leading diagonal.

Example 1

Given that a binary operation, * is defined on a set, $S = \{-2, 3, 5\}$ as $a * b = ab - (a + b)$, show whether or not * is commutative.

Solution:

$a * b = ab - (a + b), S = \{-2, 3, 5\}$

$-2 * 3 = -2(3) - (-2 + 3) = -7$

$3 * (-2) = 3(-2) - (3 + (-2)) = -7$

$-2 * 5 = -2(5) - (-2 + 5) = -13$

$5 * (-2) = 5(-2) - (5 + (-2)) = -13$

$3 * 5 = 3(5) - (3 + 5) = 7$

$5 * 3 = 5(3) - (5 + 3) = 7$

Since $a * b = b * a$ for all a and b in S, * is commutative

Example 2

A binary operation $\odot$ is defined on the set $P = \{x, y, z\}$ by the table below. Determine whether the operation $\odot$ is commutative.

$\odot$	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y

Solution

By inspection, we identify that $x \odot y = y \odot x = y$; $x \odot z = z \odot x = z$ and $y \odot z = z \odot y = x$

$\therefore \odot$ is commutative.

Alternatively, since the table is symmetric along the principal diagonal, we conclude that the operation $\odot$ is commutative.

$\odot$	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y

c. Associativity

Given that $\diamond$ is a binary operation defined on a set, S which contains a, b and c, if $a \diamond (b \diamond c) = (a \diamond b) \diamond c$, for all a, b and c in S then $\diamond$ is said to be associative.

Example 1

Given that a binary operation, * is defined on a set, $S = \{-2, 3, 5\}$ as $x * y = 2xy - 3(x + y)$, show whether or not * is associative

Solution

$x * y = 2xy - 3(x + y)$, $S = \{-2, 3, 5\}$

$-2 * 3 = 2(-2)(3) - 3((-2) + 3) = -15$

$(-2 * 3)*5 = -15 * 5$

$= 2(-15)(5) - 3((-15) + 5) = -120$

$3 * 5 = 2(3)(5) - 3((3) + 5) = 6$

$-2*(3 * 5) = -2 * 6$

$= 2(-2)(6) - 3((-2) + 6) = -36$

Since $x*(y * z) \neq (x * y)*z$, * is not associative

d. Distributive Property

Given that * and $\otimes$ are binary operations defined on the set, $S = \{a, b, c\}$ then * is distributive over $\otimes$ if $a * (b \otimes c) = (a * b) \otimes (a * c)$.

Example

If two binary operations, Δ and * are defined as $a\Delta b = 2a - 3ab$ and $a * b = -3\ b^2 + 2a$ on the set S, show whether or not Δ is distributive over *

Solution

$a\Delta b = 2a - 3ab$, $a * b = -3\ b^2 + 2a$ and S

If Δ is distributive over *, then $a\ \Delta(b * c) = (a\Delta b)*(a\Delta c)$

Let $d = b * c = -3\ c^2 + 2b$

$a\ \Delta(b * c) = a\ \Delta\ \mathrm{d}$

$= 2a - 3ad$

$= 2a - 3a(-3\ c^2 + 2b)$

$= 2a + 9a\ c^2 - 6ab$

Let $f = a\Delta b = 2a - 3ab$ and $g = a\Delta c = 2a - 3ac$

$(a\Delta b)*(a\Delta c) = f * g$

$= -3\ g^2 + 2f$

$= -3\ (2a - 3ac)^2 + 2(2a - 3ab)$

$= -3(4\ a^2 - 12\ a^2\ c + 9\ a^2\ c^2) + 4a - 6ab$

$= -12\ a^2 + 36\ a^2\ c - 27\ a^2\ c^2 + 4a - 6ab$

Since $a\ \Delta(b * c) \neq (a\Delta b)*(a\Delta c)$, Δ is not distributive over *

Theme or Focal Area: Identity and The Inverse of an Element

a. Identity element

The identity element, e of a set, S under an operation, Δ on S exists if there is an element $e \in S$ such that $a \,\Delta\, e = e \,\Delta\, a = a$. If a binary operation is not commutative, it cannot have an identity element.

Example 1

A binary operation ∇ is defined on the set R or real numbers by $p \nabla q = p + q - pq$ where p and $q \in \mathbb{R}$. Find the identity element under the operation ∇.

Solution

Check for commutativity:

$p \nabla q = p + q - pq$

$q \nabla p = q + p - qp$

Since $p \nabla q = q \nabla p$, ∇ is commutative and hence could have an identity element

Let e be the identity element, where $e \in \mathbb{R}$, then by definition, $p \nabla e = p$

But $p \nabla e = p + e - ep$

$\Longrightarrow p + e - ep = p$

$p + e(1 - p) = p$

$e(1 - p) = p - p$

$e(1 - p) = 0$

$e = 0$

Example 2

The combination table for the set $Q = \{a, b, c, d\}$ under the operation * is given below

*	a	b	c	d
a	c	a	b	d
b	a	b	c	d
c	d	c	b	a
d	b	d	a	c

State the identity element.

Solution

From the table,

$a * b = b * a = a$

$b * b = b$

$c * b = b * c = c$

$d * b = b * d = d$

$\therefore b$ is the identity element

b. Inverse of an element

The inverse of an element a of a set S under an operation Δ on S is an element $a^{-1} \in S$ such that $a \, \Delta \, a^{-1} = a^{-1} \, \Delta \, a = e$, where e is the identity element of S under the operation, Δ

Example

A binary operation ∇ is defined on the set R or real numbers by $p \nabla q = p + q - pq$ where p and $q \in R$. Find the inverse element under the operation ∇.

Solution

Let $p^{-1} \in R$ be the inverse element, then

$p \nabla p^{-1} = p + p^{-1} - p\,p^{-1} = e$

Check for commutativity:

$p \nabla q = p + q - pq$

$q \nabla p = q + p - qp$

Since $p \nabla q = q \nabla p$, ∇ is commutative and hence could have an identity element

Let e be the identity element, where $e \in \mathbb{R}$, then by definition, $p \nabla e = p$

But $p \nabla e = p + e - ep$

$\Longrightarrow p + e - ep = p$

$p + e(1 - p) = p$

$e(1 - p) = p - p$

$e(1 - p) = 0$

$e = 0$

$\Rightarrow p + p^{-1} - p\,p^{-1} = 0$

$\Longrightarrow p^{-1} = \frac{p}{1 - p}$

The inverse element, p^{-1} of p under the operation ∇ exist and it is $\frac{-p}{1-p}$

Learning Tasks

Guide learners to solve tasks related to understanding binary operations, properties, identity, and the inverse of an element. Provide support systems for learners who may encounter difficulties.

Learners are to:

a. review and explain the basic definition of binary operations

b. perform simple operations like addition and multiplication on integers and rational numbers.

c. identify and prove properties like commutativity, associativity, distributivity, and verify identity elements and inverses for elements in algebraic structures.

d. model real-world situations using abstract algebraic concepts and binary operations.

Pedagogical Exemplars

The aim of the lessons for the week is for all learners to be able solve problems involving properties of binary operations. The following pedagogical approaches are suggested for facilitators to take learners through.

a. Collaborative learning: In grouped arrangements, such as, mixed ability, mixed gender, or in pairs, engage learners in various activities to enhance their understanding of binary operations. Guide learners to recall the four basic operations and apply them to define binary operations within specified sets, subsequently solving related problems. Through think-pair-share or think-square-debate, learners discuss and determine sets suitable for defining binary operations, establishing rules governing these operations. Assist learners in delving into investigating the properties of binary operations and practice solving problems involving identity and inverse elements within these operations.

b. Experiential learning: Guide learners to actively participate in hands-on activities to create binary operations, define these operations and identify the inverse and identity elements associated with them. Through this interactive approach, guide learners to gain practical insights into the concepts of binary operations, enhancing their understanding and ability to apply these principles in various contexts.

c. Enquiry-based learning: Guide learners to use research resources (textbooks, electronic devices and any additional relevant resources) to discover applications of binary operations in real-life.

d. In a well-regulated class discussion: summarise the lesson for the week and give learners tasks to solve. Such tasks in the form of assignment or take-home tasks could be given to learners.

Key Assessment

Assessment Level 1: Recall and reproduction

1. If $a * b = a + b + 2$, where $a, b \in \mathbb{R}$, find the identity element
2. Suppose the operation * is defined on the set of real numbers $\mathbb{R}$ by $a * b = a + b + 2ab$. Find the identity element and the inverse of a under the operation *
3. The operation * is defined on the set of real numbers R by $p * q = 2p + q - 2pq$.
   a. Find
      a. $3 * -2$
      b. $3 * 5$
4. If $p * 4 = -2$ find the value of p.

Assessment Level 2: Skills and conceptual understanding

1. A binary operation ∇ is defined on the set $S = \{2,3,4,5\}$ by $p \nabla q = p + q - pq$ where p and $q \in S$.
   a. Construct the table for the operation ∇ on the set S.
   b. Determine whether or not the operation ∇ is
      i. closed under S,
      ii. commutative,
      iii. associative and
      iv. distributive over * if $a * b = ab - 3a$, $a, b \in S$
2. I am a binary operator. I can combine two elements and my result is the sum of the squares of the two-elements minus twice their product. Write a mathematical statement for this.

   Suppose the operation * is defined on the set of real numbers $\mathbb{R}$ by $a * b = a + b - 2ab$.

   a. Find
      i. $4 * 2$
      ii. $13 * -2$.
      iii. If $a * 4 = -2$ find the value of a

b. A binary operation ⊙ is defined on the set $P = \{x, y, z\}$ by the table below. Determine whether the operation ⊙ is commutative.

	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y

c. Evaluate

i. 2(3 + 4)
ii. 2(3) + 2(4)
iii. What conclusion can you draw for a) and b) about the binary operation (×)?
iv. What generalization can you make for the elements a, b and c on the sets ℚ, ℤ, 𝕎 and ℕ and the binary operations +, & and ×?
v. A binary operation ∇ is defined on the set $S = \{ 2, 3, 4, 5\}$ by $p \nabla q = p + q - pq$ where p and $q \in S$.

d. Construct the table for the operation ∇ on the set S.
e. Use your table to determine whether the operation ∇ is associative or not.
f. Use the table to investigate closure property of a set.

Assessment Level 3: Strategic reasoning

1. Given that $a * b = a^2 + b^2 - ab$ and $5 * y = 19$. Find the possible values of y
2. The table below defined by the operation θ on the set $B = \{r, s, t, u \}$

θ	r	s	t	u
r	s	u	r	t
s	u	t	s	r
t	r	s	t	u
u	t	r	u	s

3. Find, giving reasons if,
   i. B is closed under θ
   ii. θ is commutative
   iii. there is an identity element
4. Find, where possible, the inverse of the elements of set B.
   a. Given that $m * n = 3m + 2n - mn$, investigate whether the operation * has a unique identity.
   b. The combination table for the set $Q = \{a, b, c, d\}$ under the operation * is given below.

*	a	b	c	d
a	c	a	b	d
b	a	d	c	d
c	d	c	b	a
d	b	d	a	c

Determine the inverse of each element.

Assessment Level 4: Extended thinking

a. A baker produces two types of bread: wheat and potato. The wheat bread has a production cost of *GH¢* 250.00, and potato bread has a production cost of *GH¢* 300.00. The baker sells the wheat bread to retailers for GH¢ 6.00 each and the potato bread for *GH¢* 7.50 each. Last month, the baker produced 500 wheat bread and 300 potato bread.

a. Explain how you will use binary operation to find the total revenue generated from the sale of bread last month and determine the overall profit or loss.

b. Calculate the total revenue generated from the sale of bread last month and determine the overall profit or loss.

You have six shirts, two trousers and two pairs of shoes. Explain how you will use the binary operation to determine how many ways a shirt, a trouser and a pair of shoes can be worn.

Week 2

Learning Indicator(s): *Establish the properties of operations on set, including commutative, associative, and distributive, sets algebra and apply them to solve problems.*

Theme or Focal Area: Sets, Properties of Sets and Operations on Sets

A set is a group of items that share a unique description. The items that make up a set are called elements or members. Sets are represented with capital letters and can be described in using statements, rosters/ lists or rules/set builder notations.

For example, a set E described as "*the set of even numbers between 31 and 45"* (in statement form) can also be described as $E = \{32, 34, 36, 38, 40, 42, 44\}$ (in list form) $E = \{a|a \text{ is an even number}, 31 < a < 45\}$ i.e., using set builder notations read as *"E is the set of all values of a such that a is an even number and a is between 31 and 45"*. $38 \in E$ reads "38 *is a member of E"* while $46 \notin E$ is read as *"46 is not a member of E"*. The concept of sets, properties and operations on sets are applied in database systems, Venn diagram, search engine operations, network theory and in social sciences.

Types of Sets

a. Null or Empty Sets

A set which contains no element is called a null or empty set. It is represented by ϕ (phi) or $\{\}$. For example, there is no even integer between 3 and 4 hence the set, $S = \{s|s \text{ is an even integer}, 3 < s < 4\}$ is null, thus $S = \phi$. Note that $A = \{\phi\}$ is NOT a null set as it is a set which has ϕ as an element and thus, has one element.

b. Unit or Singleton Sets

A unit set is a set which has only one element. Examples: $A = \{\phi\}$, $B =$ *The set of perfect numbers between* 5 *and* 10, $C = \left\{c : c > 0,\ c = \sqrt{36}\right\}$

c. Finite Sets

If the cardinality (number of elements in a set) can be determined, then the set is said to be finite. Examples: $A = \left\{-\frac{2}{3}, 0.32, 8{,}56\%\right\}$ and $V = \{\text{All vowels in the English alphabets}\}$

d. Infinite Sets

If a set contains infinitely many members, then the set is said to be infinite. For such sets, all the elements cannot be listed. Examples: $P = \{\text{All perfect numbers}\}$ and $X = \{x : x \in \mathbb{R}, -4 < x \le 2\}$

Operations on sets

The operations on sets which will be discussed in this section are union, intersection and complements of sets.

a. Intersection of Sets

The intersection of two or more sets is the set of all members that are common to the given sets. For example, the intersection of $A = \{-6, -4, -2, 2, 4, 6\}$, $B = \{b : b \text{ is an even integer}, 3 < b \le 12\}$ and $C = \{\text{all integers that are divisible by } 2\}$ represented by $A \cap B \cap C = \{4, 6\}$

b. Union of Sets

If a set is the union set of two or more sets, then it consists of all elements that are in the individual sets and those that are common to the sets. For example, if all the elements of A, B and C are integers and $A = \{-6, -5, 0, 3, 4\}$, $B = \{b : b \text{ is an odd number}, 3 \le b < 12\}$ and

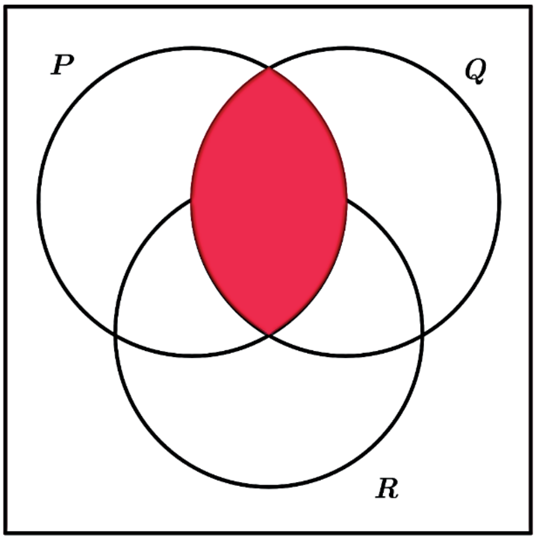
$C = \{$*positive prime numbers less than* $8\}$, then the union set of A, B and C, represented by $A \cup B \cup C = \{-6, -5, 0, 2, 3, 4, 5, 7, 9, 11\}$

c. Complements of Sets

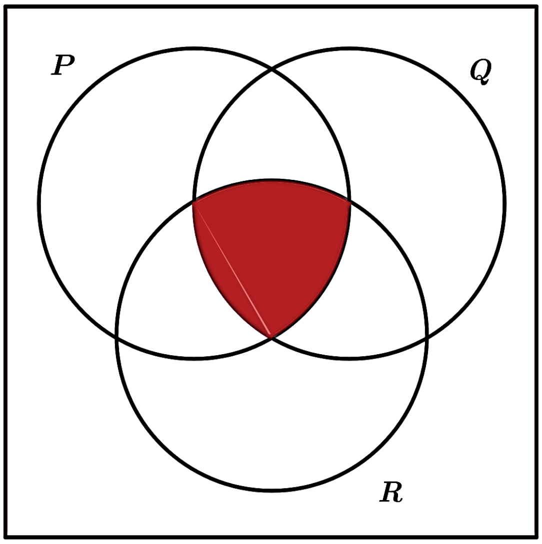
Given that set A is a subset of set S, i.e., $A \subset S$, then the complement of A, denoted by A' or $\overline{A}$ is the set that contains all elements of S which are not elements of A. For example, if $P = \{p|p$ *is a prime number*$\}$ and $P \subset \mathbb{N}$, then the complement of P, $P' = \{$*positive composite numbers*$\}$

Representing sets on the Venn diagram and solving problems using the Venn diagram.

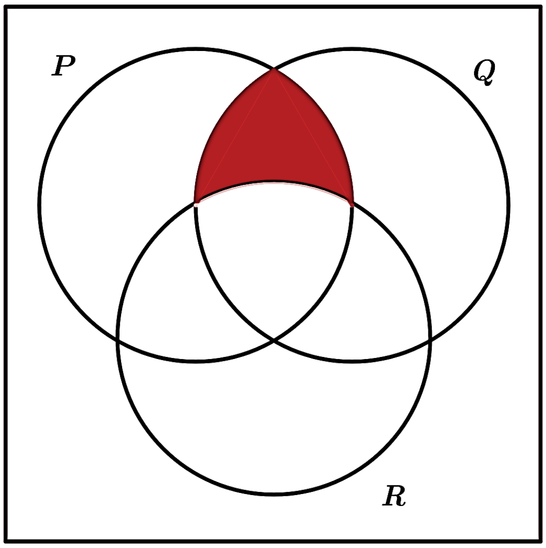
Regions of a three-set Venn diagram



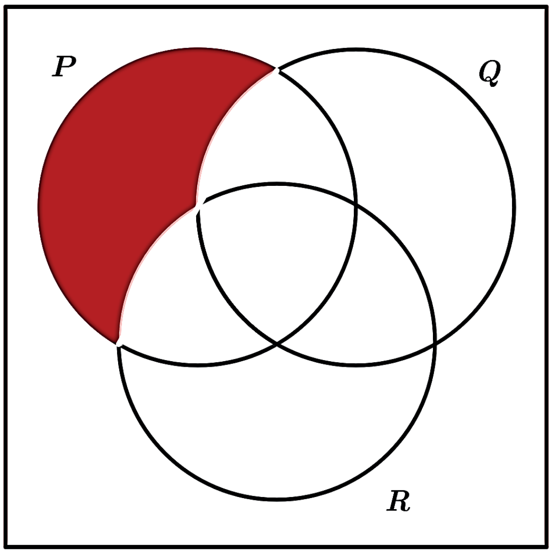
$P \cap Q$



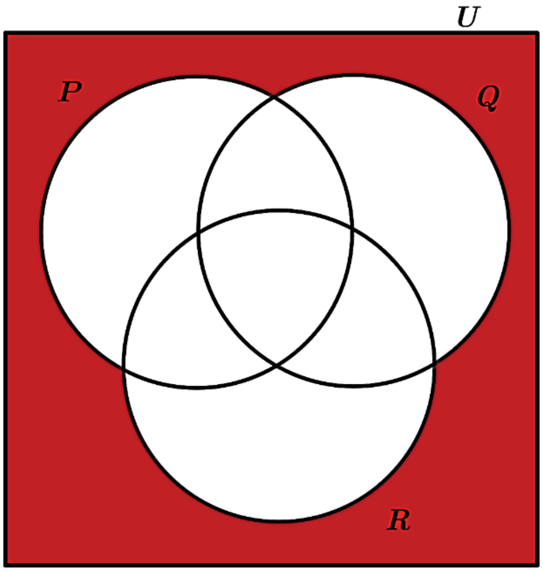
$P \cap Q \cap R$



$P \cap Q \cap R'$
(P and Q only)



$P \cap (Q \cup R)'$
(P only)



$(P \cup Q \cup R)'$

Properties of operation on sets:

a. Commutative property

i. $A \cup B = B \cup A$

ii. $A \cap B = B \cap A$

b. Associative property

i. $(A \cap B) \cap C = A \cap (B \cap C)$

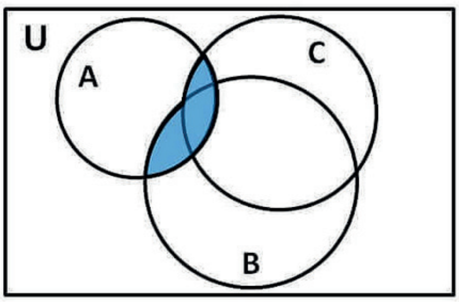
ii. $A \cup (B \cup C) = (A \cup B) \cup C$

c. Distributive property

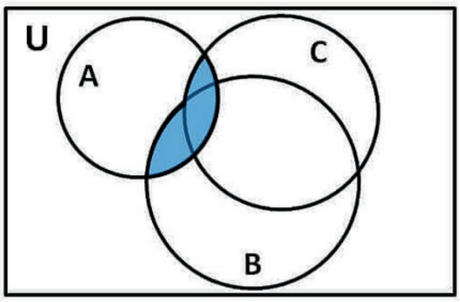
i. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

ii. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Example of illustration of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ in Venn diagram



A ∩ (B ∪ C)



(A ∩ B) ∪ (A ∩ C)

Examples

Let $U = \{1, 2, 3, \ldots, 12\}$ and $U \in \mathbb{Z}$

$A = \{x \in U : x \text{ is a prime number}\}$

$B = \{x \in U : x \text{ is an even number}\}$

$C = \{x \in U : x \text{ is divisible by } 3\}$.

Find the following sets:

a. $A \cup B$

b. $A \cap C$

c. $(A \cap C) \cup (B \cap C)$

d. $A \cap B$

Solution

$A = \{2, 3, 5, 7, 11\}$

$B = \{2, 4, 6, 8, 10, 12\}$

$C = \{3, 6, 9, 12\}$

a. $A \cup B = \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12\}$

b. $A \cap C = \{3\}$

c. $(A \cap C) \cup (B \cap C) = \{3\} \cup \{6, 12\} = \{3, 6, 12\}$

d. $A \cap B = \{2, 3, 5, 7, 11\} \cap \{2, 4, 6, 8, 10, 12\} = \{2\}$

Learning Tasks

Guide learners through tasks aimed at understanding sets, properties and operations on sets, as well as Venn diagrams involving three sets. Offer support systems to learners facing difficulties in grasping these concepts.

Learners to:

a. review and define basic sets, understand set notation

b. performs simple set operations such as union, intersection and complement.

c. construct basic Venn diagrams involving three sets

d. interprets the relationships between these sets using the diagrams.

e. accurately and analyse relationships between three sets using Venn diagrams, including identifying overlapping regions and calculating set operations based on the diagrams.

Pedagogical Exemplars

The aim of the lessons for the week is for all learners to be able to establish the properties of operations on set, including commutative, associative, and distributive, sets algebra and apply them to solve problems. The following pedagogical approaches are suggested for facilitators to take learners through.

1. **Collaborative Learning:** In convenient groups such as ability, mixed ability, mixed gender, or pairs, to engage in various activities. Guide learners to establish set identities and verify their properties, which lays a foundation for a comprehensive understanding of set theory. Assist learners to apply these set identities to solve problems across different contexts. Again, guide learners to use Venn diagrams as tools to visually represent and verify set identities, enhancing their ability to visualize and comprehend relationships between sets.

2. **Experiential Learning**: Guide learners to create set problems collectively and apply set identities, algebraic principles, and operations to solve these problems. Assist learners to se set algebra to verify relationships between three sets, denoted as Sets A, B, and C. Moreover, guide learners to create real-life scenarios involving up to three sets and apply set algebra, operations, and identities

to solve problems generated by their peers, fostering a deeper understanding of set theory and its practical applications.

3. **In a well-regulated class discussion**: summarise the lesson for the week and give learners tasks to solve. Such tasks in the form of assignment or take-home tasks could be given to learners.

Key Assessment

Assessment level 1: Recall

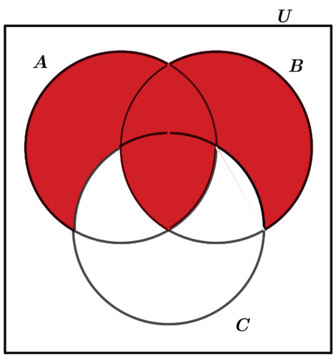
1. Consider the sets:

$A = \{red, green, blue\}$,
$B = \{red, yellow, orange\}$,
$C = \{red, orange, yellow, green, blue, purple\}$
$D = \{yellow, white\}$ *and the universal* set, $U = A \cup B \cup C \cup D$

a. find $A' \cap C$
b. find $A \cup C$ and $B' \cap A$.

2. Let $U = \{1, 2, 3, \dots, 8, 9, 10\}$ be the universal set. Consider the sets: $A = \{3, 5, 8, 10\}$, $B = \{1, 3, 4, 7, 8, 9, 10\}$ and $C = \{2, 5, 6\}$. Find each of the following:

a. $A \cup C$
b. $B \cap C$
c. $A \cap B$
d. $\overline{(A \cup C)}$
e. $(\overline{A \cap B}) \cup C)$
f. Write an expression for the shaded region.

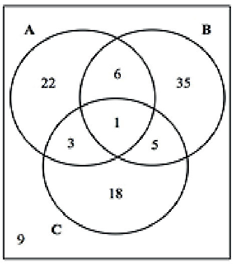


3. Suppose the universal set is $U = \{all\ odd\ numbers\ from\ 1\ to\ 15\}$ and $A = \{1, 3, 9, 11\}$, then list the elements of A'

4. A, B and C are three intersecting subsets of the universal set ξ. Draw a Venn diagram and shade the following regions:

a. A
b. C
c. B
d. $A \bigcap C$
e. $B \bigcup C$
f. $(B \bigcap C) \bigcup A$

5. Consider the Venn diagram below. The numbers indicate the number of elements in the regions



Determine the following

a. $n(A)$
b. $n(C)$
c. $n(B)$
d. $n(A \cap C)$
e. $n(B \cup C)$
f. $n((B \cap C) \cup A)$

Level 2 Skills of conceptual understanding:

The set $A = \{1, 3, 5\}$. What is a larger set that this might be a subset of?

1. Draw a Venn Diagram to represent that set G is a proper subset of set P.

2. A pharmaceutical company is considering manufacturing new toothpaste. They are considering two charcoal flavours, strawberry and mint. In a sample of 74 people, it was found that 45 liked strawberries, 37 liked mint and 21 liked both types

3. Create a Venn diagram to model the information.

4. How many liked only strawberry?

5. How many liked only mint?

6. How many liked exactly one of the two (that is they liked one but not the other)?

Level 3 Strategic reasoning:

a. A survey asks: "Which online social media have you used in the last month: WhatsApp, Facebook, Both". The results show 42% of those surveyed have used WhatsApp, 70% have used Facebook, and 20% have used both. How many people have used neither WhatsApp nor Facebook?

b. Find $(A \cup B)'$ if $n(A) = 42$, $n(B) = 70$ and $n(A \cap B) = 20$

c. In a survey of 115 pet owners, 26 said they own a dog, and 64 said they own a cat. 5 said they own both a dog and a cat. Use a Venn diagram to determine how many of the pet owners surveyed owned neither a cat nor a dog?

Level 4 Extended critical thinking and reasoning:

1. Write an expression for the regions outlined in red

2. A survey asked people what alternative transportation modes they use. 30% use the bus; 20% ride a bicycle, 25% walk; 5% use the bus and ride a bicycle 10% ride a bicycle and walk; 12% use the bus and walk; 2% use all three. Use the data to complete a Venn diagram, then determine:

a. What percent of people only ride the bus
b. How many people don't use any alternate transportation

Week 3

Learning Indicator(s):

a. *Expand Binomial expressions for positive integer indices using Pascal's triangle*

b. *Use the combination approach and other approaches to determine the coefficient and exponent of a given term in an expansion*

Theme or Focal Area: The concept of binomial theorem

A binomial is a simplified polynomial with two terms i.e., of the form $(p + q)$. $(2x^2 + 3)$, $(y - 2x)$ and $(-5 + 0.2x)$ are all examples of binomials since they have exactly two terms only but $-\frac{1}{4}y^3$, $(2x^2 + 3y - 4)$ and $(y + 3 - 0.1x)$ are not binomials since in their simplest forms, do not have exactly two terms.

In certain situations, in mathematics, it is necessary to write $(p + q)^n$ as the sum of its terms. Because $p + q$ is a binomial, this process is called expanding the binomial. For small values of n, it is relatively easy to write the expansion by using multiplication. For example, $(p + q)^2$ can be written as $(p + q)(p + q)$ and expanded to obtain $p^2 + 2pq + q^2$ and $(p + q)^3$, written as $(p + q)^2(p + q)$ and so on. We could continue to build on previous expansions and eventually have quite a comprehensive list of binomial expansions. Instead, we could look for a theorem that will enable us to expand directly. This theorem is the binomial theorem. This theorem is applied in probability and statistics, genetics and biology, finance and investment, engineering and physics and computer science and Information Technology.

Theme or Focal Area: Binomial expansion and Pascal's triangle

Consider the expansion of $(p + q)^n$ in the table for positive integer values of n

Binomial	Expansion	Coefficients
$(p+q)^0$	1	1
$(p+q)^1$	$p+q$	1 1
$(p+q)^2$	$(p+q)(p+q)$ $= p^2 + 2pq + q^2$	1 2 1
$(p+q)^3$	$(p+q)\,(p+q)^2$ $= (p+q)(p^2 + 2pq + q^2)$ $= p^3 + 3p^2q + 3pq^2 + q^3$	1 3 3 1
$(p+q)^4$	$(p+q)\,(p+q)^3$ $= (p+q)(p^3 + 3p^2q + 3pq^2 + q^3)$ $= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$	1 4 6 4 1
$(p+q)^5$	$(p+q)\,(p+q)^4$ $= (p+q)(p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4)$ $= p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$	1 5 10 10 5 1

The following observations can be made from the expansions $(p + q)^n$ for $n = 1, 2, 3, 4, 5$

a. The first term is p^n

b. The exponent on p (the first term of the binomial) decreases by 1 for each successive term

c. The exponent on q (the second term of the binomial) increases by 1 for each successive term

d. The last term is q^n

e. The degree of each term is n

Writing the coefficients alone in a triangular pattern gives Pascal's triangle, as shown in the last column of the table which was named in honour of the 17th-century mathematician Blaise Pascal (1623-1662)

The following observation can also be made from Pascal's triangle:

a. The first and last terms of each row are 1

b. Each number in the interior of the triangle is the sum of the two numbers just above it (one to the right and one to the left).

For example, in the sixth row from the top, 5 (the second coefficient) is the sum of 1 and 4, the first and second coefficients respectively in the fifth row, 10 is the sum of 4 and 6, and so on. To obtain the coefficients for $(p + q)^6$ we attach the seventh row to the table by starting and ending with 1, and adding pairs of numbers from the sixth row.

1st Row							1							
2nd Row						1		1						
3rd Row					1		2		1					
4th Row				1		3		3		1				
5th Row			1		4		6		4		1			
6th Row		1		5		10		10		5		1		
7th Row	1		6		15		20		15		6		1	

We then use these coefficients to expand as

$(p + q)^6 = p^6 + 6\,p^5\,q + 15\,p^4\,q^2 + 20\,p^3\,q^3 + 15\,p^2\,q^4 + 6p\,q^5 + q^6.$

NB: The Pascal's triangle defines the coefficients which appear in binomial expansions with positive powers. That means the nth row of Pascal's triangle comprises the coefficients of the expanded expression of the polynomial $(p + q)^n$

Example 1

Use Pascal's triangle to expand $(a + b)^5$

Solution

Number of terms = $n + 1 = 5 + 1 = 6$ terms Therefore, there will be 6 terms involving a^5, $a^4\,b$, $a^3\,b^2$, $a^2\,b^3$, $a\,b^4$, b^5 each with degree 5. From Pascal's triangle, the coefficients are respectively: 1, 5, 10, 10, 5 and 1

The expansion of $(a + b)^5$ in descending powers of a is

$(a + b)^5 = a^5 + 5\,a^4\,b + 10\,a^3\,b^2 + 10\,a^2\,b^3 + 5a\,b^4 + b^5$

Example 2

Expand $\left(\frac{1}{x} - 2\sqrt{x}\right)^5$

Solution

From the expansion of $(a + b)^5$ in the previous example and by comparison i.e., $a = \frac{1}{x}$ and $b = -2\sqrt{x}$,

$$\left(\frac{1}{x} - 2\sqrt{x}\right)^5 = \left(\frac{1}{x}\right)^5 + 5\left(\frac{1}{x}\right)^4(-2\sqrt{x}) + 10\left(\frac{1}{x}\right)^3(-2\sqrt{x})^2 + 10\left(\frac{1}{x}\right)^2(-2\sqrt{x})^3 + 5\left(\frac{1}{x}\right)(-2\sqrt{x})^4 + (-2\sqrt{x})^5$$

$$= \frac{1}{x^5} - \frac{10\sqrt{x}}{x^4} + \frac{40}{x^2} - \frac{80\sqrt{x}}{x} + 80x - 32\sqrt{x^5}$$

$$= x^{-5} - 10x^{-\frac{7}{2}} + 40x^{-2} - 80x^{-\frac{1}{2}} + 80x - 32x^{\frac{5}{2}}$$

Example 3

Use Pascal's triangle of binomial expansion to obtain the value of $(2.007)^5$, correct to six decimal places

Solution:

$(2.007)^5 = (2 + 0.007)^5$

$= 2^5 + 5(2^4)(0.007) + 10(2^3)(0.007^2) + 10(2^2)(0.007^3) + 5(2)(0.007^4) + 0.007^5$

$\approx 32 + 80(0.007) + 80(0.000049)$

the other terms are cut off since they result in numbers having more than 6 decimal places

≈ 32.563920 (6 *dp*)

Theme or Focal Area: Combination Approach

It is not practical to use Pascal's triangle to expand binomial expressions with larger positive integer values as exponents since the method is recursive. It requires that to obtain the coefficients of the expansion of $(p + q)^n$, we would need to find the coefficients of the expansion of $(p + q)^{n-1}$. Thus, to find the 100th row of the Pascal's triangle, we must first find the preceding 99 rows. The coefficients in the Pascal's triangle however, can be generated using the combination approach.

The number of combinations of n objects by selecting r at a time, written as ${}^nC_r$ or $\binom{n}{r}$ can be defined as ${}^nC_r = \frac{n!}{(n-r)!r!}$ and $n! = n(n-1)(n-2)(n-3)\ldots(3)(2)(1)$

The Pascal's triangle can be generated with combinations as such

Power.

0	1st	${}^0C_0 = 1$					
1	2nd	${}^1C_0 = 1$	${}^1C_1 = 1$				
2	3rd	${}^2C_0 = 1$	${}^2C_1 = 2$	${}^2C_2 = 1$			
3	4th	${}^3C_0 = 1$	${}^3C_1 = 3$	${}^3C_2 = 3$	${}^3C_3 = 1$		
4	5th	${}^4C_0 = 1$	${}^4C_1 = 4$	${}^4C_2 = 6$	${}^4C_3 = 4$	${}^4C_4 = 1$	
5	6th	${}^5C_0 = 1$	${}^5C_1 = 5$	${}^5C_2 = 10$	${}^5C_3 = 10$	${}^5C_4 = 5$	${}^5C_5 = 1$

If n is a positive integer, the expansion $(a + b)^n$ is given by:

$$(a+b)^n = \left[{}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_n\, a^{n-n} b^n\right]$$

This is the binomial theorem and the following conclusions can be made:

a. The total number of terms in the expansion $(a + b)^n$ is $(n + 1)$

b. The sum of exponents of a and b is always n.

c. The number of terms in the expansion of $(a + b)^n + (a - b)^n$ is $\frac{n + 2}{2}$ if "n" is even or $\frac{n + 1}{2}$ if "n" is odd.

d. The number of terms in the expansion of $(x + a)^n - (x - a)^n$ is $\left(\frac{n}{2}\right)$ if "n" is even or $\frac{n + 1}{2}$ if "n" is odd.

NB: Investigations should be conducted to establish the conclusions given

Examples

Use the binomial theorem to write out the first five terms of the binomial expansion and simplify $(x+2y)^{20}$

Solution

$(x+ 2y)^{20} = x^{20} + \binom{20}{1} x^{19} (2y)^1 + \binom{20}{2} x^{18} (2y)^2 + \binom{20}{3} x^{17} (2y)^3 + \binom{20}{3} x^{16} (2y)^4 + \ldots$

Technology Tip (use of calculator evaluation)

$\binom{20}{1} = \frac{20}{1} = 20 \; \binom{20}{2} = \frac{20 \,.\, 19}{1 \,.\, 2} = 190$

$\binom{20}{3} = 2\frac{0 \,.\, 19 \,.\, 18}{1 \,.\, 2 \,.\, 3} = 1140$

$\binom{20}{3} = \frac{20 \,.\, 19 \,.\, 18 \,.\, 17}{1 \,.\, 2 \,.\, 3 \,.\, 4} = 4845$

Therefore, the first five terms of $(x+ 2y)^{20}$ are

$x^{20} + 20 \,.\, x^{19} (2y)^1 + (190) \,.\, 4 \,.\, x^{18} y^2 + (1140) \,.\, 8 \,.\, x^{17} y^3 + 4\,845 \,.\, 16 \,.\, x^{16} y^4$

$= x^{20} + 40 \,.\, x^{19} y^2 + 760 \,.\, x1^8 y^4 + 9120 \,.\, x^{17} y^6 + 77520 \,.\, x1^6 y^8$

NB: Support learners to apply Calculator Evaluation (i.e, Use Technology Tips to evaluate the expression $(1 + x)^{10}$

Learning Tasks

Guide learners through tasks aimed at understanding defining and expanding binomial expressions and applying binomial expansions. Offer support systems to learners facing difficulties in grasping these concepts, ensuring comprehensive understanding and successful application of these mathematical principles.

Learners to:

a. review and explain the basic concept of binomial expressions and expansions.

b. identify terms like coefficients, variables, and exponents within binomial expansions.

c. apply binomial expansions to solve basic mathematical problems, such as finding coefficients or simplifying expressions with binomial terms.

d. engage in group activities to evaluate powers of decimal numbers using binomial expansions.

PEDAGOGICAL EXEMPLARS

The aim of the lessons for the week is for all learners to be able to Expand Binomial expressions for positive integer indices using Pascal's triangle and use the combination approach and other approaches to determine the coefficient and exponent of a given term in an expansion. The following pedagogical approaches are suggested for facilitators to take learners through.

1. **Collaborative Learning:** In groups conveniently based on ability, mixed ability, mixed gender or pairs, guide learners to establish both personal and conventional strategies for determining coefficients of terms in a binomial expansion with positive integer indices across different contexts. Assist learners to explore combination strategies for determining various terms of a binomial expansion with positive integer indices, enhancing their understanding and application of binomial expansions in diverse mathematical scenarios.

2. **Experiential Learning:** Guide learners to collaboratively engage in hands-on activities, focusing on learning through practical experience, to create binomial expressions with positive integer indices. Assist learners to apply both personal and conventional strategies to determine the coefficients of the terms in their expansions. Additionally, facilitates learners' group work as they work together to establish an understanding of the binomial theorem, developing their proficiency in handling binomial expressions and applying mathematical concepts effectively.

3. **In a well-regulated class discussion**: summarise the lesson for the week and give learners tasks to solve. Such tasks in the form of assignment or take-home tasks could be given to learners.

Key Assessment

Assessment Level 1: Recall

a. Indicate whether the following statement is true or false.

i. Pascal's triangle is a triangular array of numbers in which the first and the last numbers in each row are 1

ii. In Pascal's triangle, each number is just the two numbers above it added together.

b. Provide appropriate responses (True or False) to the following tasks

i. For every positive integer n, $(3n)! = (3)!(n)!$

ii. There are ten terms in the expression $(1-x)^{10}$

iii. The middle term of the expansion of $\left(1+\frac{1}{x}\right)^8$ is 70.

c. Use the combination formula to expand the following:

i. $(a+b)^2$

ii. $(a+b)^4$

d. Obtain the expansion of $\left(2x+\frac{1}{2}\right)^4$ in descending powers of x

Assessment Level 2: Skills of conceptual understanding:

1. Use Pascal's triangle to expand the following:

i. $(a+b)^2$

ii. $(a+b)^4$s

2. Obtain the expansion of $\left(2x-\frac{1}{2}\right)^4$ in descending powers of x

3. Obtain the expansion of $(6x^3 - 5y^2)^4$

4. Find the coefficient of the term $(x)^{10}$ in the binomial expansion of the expression: $(1+x)^{25}$

5. Find the coefficient of the term $x^9 y^5$in the binomial expansion of the expression: $(2x + 3y)^{14}$
6. Find the tenth term in the expansion of $(x + 3)^{12}$.
7. Find the coefficient of the term x^{10} in the binomial expansion of the expression: $\left(2x+\frac{1}{x}\right)^{25}$

Assessment level 3: Strategic reasoning;

1. a. Find the expansion of $(a + b)^6$
   b. If $a = \frac{1}{x}$ and $b = -2\sqrt{x}$, find the expression involving x^0
2. Find the expression for the 4th term of the expression $\left(\frac{1}{x} - 2\sqrt{x}\right)^6$
3. Using the terms in the expansion of $(x + y)^5$, find $\sum(x + y)^5$, if $x = 1$ and $y = 0.05$

Assessment level 4: Extended critical thinking and reasoning:

a. Write down the binomial expansion of $(a + b)^4$ and use your expansion to evaluate $(2.01)^4$.

b. Find the remainder when 7^{103} is divided by 25.

c. If the fractional part of the number $\left(\frac{2^{403}}{15}\right)$ is $(K \_ 15)$, then find K

d. Show that $11^9+ 9^{11}$ is divisible by 10.

Section 1 Review

This section reviews all the lessons taught for the first three (3) weeks. It is a summary of what the learner should have learnt. These first three weeks provided a strong foundation for your journey into algebra. We explored sets, a fundamental concept for organizing and classifying objects, essential for various fields like statistics and computer science. We delved into binary operations, the workhorses of mathematical calculations, and analysed their properties to understand their behaviour within different sets. Finally, we introduced the Binomial Theorem, a powerful tool for expanding binomial expressions, with applications spanning finance, physics, and computer science.

The following learning resources are recommended to facilitate teaching and learning: **Teaching/Learning Resources:** Maths posters, White board Pan balance, Videos, Mini whiteboards or laminated white paper, Dry erase markers and erasers, Algebraic tiles, Patterns, Calculator, Paper grids, Technological tools such as computer, mobile phone, you tube videos etc.

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SECTION 2: SURDS, INDICES AND LOGARITHMS

Strand: **Modelling with Algebra**

Sub-Strand: Number and Algebraic Patterns

Content Standard: Demonstrate knowledge and understanding of numbers about Surds, Indices and Logarithms

Learning Outcome(s):

a. *Perform basic operations on surds*

b. *Solve simple indicial and logarithmic equations*

INTRODUCTION AND SECTION SUMMARY

The concept of exponents has roots in ancient civilizations like Mesopotamia and Egypt. It gained a formal structure in the 17th century with mathematicians like René Descartes. Basic properties of indices, such as product rule ($a^m * a^n = a^{(m+n)}$), quotient rule $\left(\frac{a^m}{a^n} = a^{(m-n)}\right)$ and power of a power rule ($a^{(mn)} = (a^m)^n$), will be explored. These will be crucial for simplifying expressions and performing calculations efficiently. Surds (radicals) are expressions containing roots (like square root or cube root) that cannot be simplified further. We will explore simplifying techniques for surds, including rationalizing the denominator and applying the product and quotient properties of radicals. We will learn how to perform addition, subtraction, multiplication, and division on expressions containing radicals. Understanding surds is essential for solving various equations in algebra, particularly those involving irrational solutions. Logarithms are the inverse operation of exponentiation. They help us find the exponent to which a base number must be raised to get another specific number. For example, $log_2(8) = 3$, because $2^3 = 8$. We will introduce basic properties of logarithms, such as $log(a * b) = log(a) + log(b)$ and $log(a^n) = n * log(a)$.

The weeks covered by the section are:

Week 4:

1. Definition, properties and simplification of surds
2. Rationalisation of surds

Week 5:

1. Simplification of expressions (surds and indices)
2. Indicial equations
3. Definition and relationship between indices and logarithms and logarithmic equations

SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in practical activities on Surds, Indices and Logarithms. Learners should be given the platform to work in groups to develop their own real-life questions and find answers. Therefore, Experiential learning activities and Mixed-ability groupings should dominate the lessons on these concepts. All learners, irrespective of their learning abilities should be assisted to take part fully in investigations and presentation of findings. However, make considerations and accommodations for the different groups. That is, offer approaching proficiency learners the opportunity to make oral presentations. Then, extend activities for the above average/ highly proficient learners to use formulae and computer applications to solve problems.

ASSESSMENT SUMMARY

Assessment methods ranging from quizzes, tests, and homework assignments can be used to evaluate learners understanding of concepts and their ability to solve problems. Performance tasks like solving real-world problems involving Surds, Indices and Logarithms will also be used to assess learner's application of these mathematical skills. Also, make use of various visual aids and charts on graphs and interactive mapping tools to engage learners in hands-on learning experiences. Assessment strategies which vary from Level 1 to Level 4 questions of the DoK will be used. Teacher should record the performances of learners for continuous assessment records.

WEEK 4

Learning Indicator(s):

1. *Investigate the properties of surds and perform basic arithmetic operations on surds*
2. *Rationalise surds with binomial denominators*

Theme or Focal Area: Definition, properties and simplification of surds

Surds were discovered and defined by a European mathematician, Gherardo of Cremona, in 1150 BC (Joseph, 2010). He used the Pythagoras theorem to find the diagonal of a square and the value of the first surd. He termed this value 'voiceless' because the root value had no meaning at that time. Surds are numbers with roots that cannot be simplified to whole numbers. They are square roots, or other roots, that cannot be written as a simple fraction. Surds or radical expressions contain roots (like $\sqrt{2}$) that are not whole numbers. For example, $\sqrt{2} = 1.4142135624$ $\sqrt{3} = 1.7320508076$, $\sqrt{35} = 5.9160797831$ etc. However, $\sqrt{25} = 5$ and $\sqrt{81} = 9$, both have whole number solutions, thus, are not surds.

Types of Surds:

a. Pure Surds: A surd having only a single irrational number is called a pure surd. Example., $\sqrt{7}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{2}$
b. Mixed Surds: A surd having a mix of a rational number and an irrational number is called a mixed surd. Example, $5\sqrt{3}$,
c. Compound Surds: A surd composed of two surds or a surd and a rational number is called a compound surd. Example, $\sqrt{3} + \sqrt{10}$, $3 + \sqrt{7}$,
d. Binomial Surd: When two surds give rise to one single surd, the resultant surd is known as a binomial surd. Example, $\sqrt{30} = \sqrt{15 \times 2}$

Rules/Properties of surds;

a. $\sqrt{a} \times \sqrt{a} = (\sqrt{a})^2 = a$
b. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
c. $a \times \sqrt{b} = a\sqrt{b}$
d. $a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$
e. $a\sqrt{b} \times c\sqrt{b} = ac\left(\sqrt{b}\right)^2 = acb$
f. $\sqrt{(a \times b)} = \sqrt{a} \times \sqrt{b} \quad a, b \geq 0$
g. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad b > 0$
h. $\sqrt[n]{a} = a^{\frac{1}{n}}$
i. $a\sqrt{c} + b\sqrt{c} = (a + b)\sqrt{c}$
j. $a\sqrt{c} - b\sqrt{c} = (a - b)\sqrt{c}$
k. $\frac{a}{a + \sqrt{b}} = \frac{a^2 - a\sqrt{b}}{a^2 - b}$
l. $\frac{a}{a - \sqrt{b}} = \frac{a^2 + a\sqrt{b}}{a^2 - b}$

m. $\sqrt{a} + \sqrt{b} \neq \sqrt{(a + b)}$

n. $\sqrt{a} - \sqrt{b} \neq \sqrt{(a - b)}$

Simplification of surds

Examples 1

Simplify $\sqrt{108}$

Solution

$\sqrt{108} = \sqrt{36} \times \sqrt{3} = 6\sqrt{3}$

Example 2

What is the square root of $(8 + 2\sqrt{15})$?

Solution

$(8 + 2\sqrt{15}) = (5 + 3) + 2(\sqrt{3} \times \sqrt{5})$

Thus, root of $(8 + 2\sqrt{15}) = \sqrt{(5 + 3) + 2(\sqrt{3} \times \sqrt{5})}$

$= \sqrt{(\sqrt{5})^2 + (\sqrt{3})^2 + 2(\sqrt{3} \times \sqrt{5})})$

Comparing $= \sqrt{(\sqrt{5})^2 + (\sqrt{3})^2 + 2(\sqrt{3} \times \sqrt{5})})$ to the identity

$(a + b)^2 = a^2 + b^2 + 2ab$

For example, to expand $(x + 2)^2 = (x + 2)(x + 2)$

$= x(x + 2) + 2(x + 2)$

$= x^2 + 2x + 2x + 4$

$= x^2 + 4x + 4$

Thus, we have $\sqrt{(\sqrt{5} + \sqrt{3})^2} = \sqrt{5} + \sqrt{3}$

Therefore, the root of $(8 + 2\sqrt{15}) = \sqrt{5} + \sqrt{3}$

NB:

1. *Unlike surds cannot be added or subtracted but similar surds can be added or subtracted.*
2. *Like and unlike surds can be*
   a. *multiplied.*
   b. *Divided.*
   c. *written in exponential form.*

Conjugate of surds

Conjugate surds are like mirror images in the world of surds. They are formed by simply changing the sign of the radical part in a surd expression. For example, the conjugate of $\sqrt{2+3}$ is $\sqrt{2-3}$. When you multiply a surd by its conjugate, the result is a rational expression due to the difference of squares property, making them helpful for simplifying expressions containing surds.

The table below lists various surds and their conjugates.

Surd	Conjugate
$\sqrt{a}$	$\sqrt{a}$
$a\sqrt{b}$	$\sqrt{b}$
$a+\sqrt{b}$	$a-\sqrt{b}$
$a-\sqrt{b}$	$a+\sqrt{b}$
$a+b\sqrt{c}$	$a-b\sqrt{c}$
$a-b\sqrt{c}$	$a+b\sqrt{c}$
$\sqrt{a}+\sqrt{b}$	$\sqrt{a}-\sqrt{b}$
$\sqrt{a}-\sqrt{b}$	$\sqrt{a}+\sqrt{b}$

Theme or Focal Area: Rationalisation of surds

Rationalising the denominator of surds means making the denominator of a fraction a rational number by multiplying it by a suitable factor. If the denominator is a single surd, the factor is the same surd. If the denominator is a binomial expression with a surd, the factor is the same expression with the opposite sign in the middle. The numerator of the fraction is also multiplied by the same factor. The answer is then simplified.

Examples

Rationalising surds involves changing the surd denominator to a rational number. That is:

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}}$$

$$= \frac{a\sqrt{b}}{b}$$

Sometimes the denominator is a binomial. For instance, to rationalize $\frac{1}{3-\sqrt{5}}$, we first multiply the given surd by its conjugate and then simplify below as:

$$\frac{1}{3-\sqrt{5}} = \frac{1}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}$$

$$= \frac{3+\sqrt{5}}{3-\sqrt{3}(3+\sqrt{5})}$$

$$= \frac{3+\sqrt{5}}{9-5} = \frac{3+\sqrt{5}}{4}$$

Learning Tasks

Guide learners to perform the following task to check for understanding. Support system should be offered to learners who will struggle.

Learners to

a. review and investigate the properties of surds.

b. perform basic arithmetic operations on surds

c. rationalise surds with binomial denominators

PEDAGOGICAL EXEMPLARS

The objectives for this week's lessons includes helping learners to be able to investigate the properties of surds and perform basic arithmetic operations on surds. Learners will be able to rationalise surds with binomial denominators. The following pedagogical strategies are recommended in the curriculum:

Collaborative Learning: Learners will be working in convenient groups (ability, mixed ability, mixed gender, or pairs etc.) to:

1. explore the basic properties and operations on surds in different contexts.
2. discuss the various properties of surds using participatory activities such as **think-pair-share/square and debate**.
3. create surds problems involving basic operations, properties and types of surds using participatory activities such as **think-pair-share/square and debate**
4. solve problems involving rationalization

Experiential Learning: Learners engage in hands-on activity (learning by doing) to

1. create surds problems involving basic operations, properties and types of surds
2. discuss when and how to rationalise surds

Enquiry based learning: Learners use research resources (textbooks, electronic devices and any additional relevant resources) to deepen their understanding of operations, properties and types of surds as well as rationalisation of surds.

KEY ASSESSMENT

Assessment level 1: Recall

Put the following surds in their simplest form where possible. For those that cannot be further simplified, state the reasons why.

a. $\sqrt{5} + \sqrt{7}$

b. $3\sqrt{2} + 5\sqrt{2}$

c. $\sqrt{7} - \sqrt{5}$

d. $3\sqrt{2} - 5\sqrt{2}$

e. $3\sqrt{2} \times 5\sqrt{2}$

f. $\sqrt{15} \div \sqrt{5}$

g. Simplify the following surd expression: $\sqrt{12} + \sqrt{27}$

h. Solve for x : $2\sqrt{3x + 5} = 4\sqrt{x + 1}$

i. Given that $\sqrt{a} + \sqrt{b} = 7$ and $\sqrt{a} - \sqrt{b} = 1$, find the value of a and b.

j. Rationalize the denominator of the fraction: $\frac{1}{\sqrt{5}}$

k. Simplify and rationalize the expression: $\frac{2+\sqrt{6}}{\sqrt{2}}$

l. Rationalize the denominator of the expression: $\frac{(\sqrt{3}+\sqrt{7})}{(\sqrt{7}+3)}$

Assessment Level 2: Skills and Conceptual Understanding

1. Calculate $\sqrt{0.9}$
2. which of the following statements is/are true?
   a. $2\sqrt{3} > 3\sqrt{2}$,
   b. $4\sqrt{2} > 2\sqrt{8}$
3. What is the conjugate of
   a) $1+\sqrt{3}$
   b) $8\sqrt{5}+6$

Assessment Level 3: Strategic Reasoning

1. What is the square root of $(10+\sqrt{25})(12-\sqrt{49})$?
2. If $(3+2\sqrt{5})^2 = 29 + k\sqrt{5}$, find the value of k
3. Find the value of y, if $\sqrt{64} - 3\sqrt{64} = -4\sqrt{y}$, where $y > 0$,

Assessment Level 4: Extended Thinking

1. Simplify $\frac{\sqrt{48}+2\sqrt{27}}{\sqrt{12}}$, given your answer as an integer
2. Express $\frac{1-5\sqrt{5}}{3+\sqrt{5}}$ in the form $m + n\sqrt{5}$; $m, n \in Z$
3. Show that $\frac{\sqrt{75}+\sqrt{27}}{\sqrt{3}}$ is an integer and find its value
4. Show that $\frac{x-25}{\sqrt{x}+5} = \sqrt{x} - 5$
5. Rationalize the denominator of $\frac{8}{1+2\sqrt{3}}$
6. Simplify $\frac{8}{1+2\sqrt{3}} \times 8\frac{}{-2\sqrt{3}}$

WEEK 5

Learning Indicator(s):

1. *Recollect the initial laws of indices and establish other laws for negative powers and roots*
2. *Recognise the relationship between surds and indices and apply laws of indices to simplify expressions*
3. *Pose and solve simple equations involving indices*
4. *Establish the relationship between indices and logarithms and use the properties of logarithms to solve related problems in one base*

Theme or Focal Area: Definition and laws of indices

Index is the power or exponent of a number or a variable. The plural for index is **indices.** For instance, in the expression, 3^4, 3 is called the **base** and 4 is called the **index, power** or **exponent** and it is read as ***"three exponent four"*** or *"**three to the power 4**"*. The concept of indices is widely applied in simplifying mathematical expressions, algebraic manipulation, writing scientific notation, differentiation and integration, geometry and trigonometry, compound interest calculation, etc.

Example

Write the following single numbers as exponents in their simplest forms.

a. 16

b. 27

c. 100

Solutions:

a. Least prime factor of 16 = {2}

$16 = 2 \times 2 \times 2 \times 2 = 2^4$

b. Least prime factor of 27 = {3}

$27 = 3 \times 3 \times 3 = 3^3$

c. Prime factors of 100 = {2, 5}

$100 = 10 \times 10 = 2 \times 5 \times 2 \times 5 = 2^2 \times 5^2$

Verification of rules of indices

i. $a^m \times a^n = a^{m+n}$, and by extension, once the base (a) is equal $a^m \times a^n \times a^p \times \ldots = a^{m+n+p+\ldots}$

ii. $a^m \div a^n = a^{m-n}$

iii. $(a^m)^n = a^{mn}$, and by extension, $(a^m \times b^m \times \ldots)^n = a^{mn} \times b^{mn} \times \ldots$

vi. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

v. $\left(\frac{a \times b \times \ldots}{c \times d \times \ldots}\right)^m = \frac{a^m \times b^m \times \ldots}{c^m \times d^m \times \ldots}$

vi. $(ab)^m = a^m\, b^m$

vii. $\left(\frac{a^m}{b^m}\right)^n = \frac{a^{mn}}{b^{mn}}$

viii. $\left(\frac{a^m \times b^m \times \ldots}{c^m \times d^m \times \ldots}\right)^n = \frac{a^{mn} \times b^{mn} \times \ldots}{c^{mn} \times d^{mn} \times \ldots}$

ix. $a^{\frac{1}{n}} = \sqrt[n]{a}$

Other rules are:

1. Negative exponent Rule, that is $a^{-m} = 1\overline{a^m}$ by extension $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$
2. Zero power rule, that is

   $a^0 = 1$, where $a \neq 0$

Example

Simplify $8^{\frac{2x}{3}}$

Solution:

$8^{\frac{2x}{3}} = \left(\sqrt[3]{8}\right)^{2x} = 2^{2x} = 4^x$

Theme or Focal Area: Simplification of expressions (surds and indices)

As already explained, surds are the root values that cannot be written as whole numbers. More so, indices are the exponents of a value. Thus, given 2^5, 5 is the index while 2 is the base. Taking a square root is the inverse process of squaring.

Solving indicial problems involving surds

Example

If $a = 3 - \sqrt{3}$, show that $a^2 + \frac{36}{a^2} = 24$

Solution:

Substituting $a = 3 - \sqrt{3}$ into $a^2 + \frac{36}{a^2}$

$$\Rightarrow a^2 + 3\frac{6}{a^2} = (3 - \sqrt{3})^2 + 3\frac{6}{(3 - \sqrt{3})^2}$$

$$= 9 - 6\sqrt{3} + 3 + \frac{36}{9 - 6\sqrt{3} + 3}$$

$$= 12 - 6\sqrt{3} + \frac{36}{12 - 6\sqrt{3}}$$

$$= 12 - 6\sqrt{3} + \frac{6}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$

$$= 12 - 6\sqrt{3} + \frac{12 + 6\sqrt{3}}{4 - 3}$$

$$= 24$$

Theme or Focal Area: Indicial equations

Areas under indicial equations to be explored include;

- **a.** Solving simple indicial or exponential equations
- **b.** Solving simultaneous equations involving exponents or indices
- **c.** Application of exponential indicial equations (such as growths and/or decays)
- **d.** Solve logarithm problems in one base.
- **e.** Change of base
- **f.** Using Logarithms to solve exponential equations

Example 1

Solve the equation $2^x = 16$

Solution:

From $2^4 = 16$

$\Rightarrow 2^x = 2^4$

$x = 4$

Example 2

Given that $y = 2x$ and $3^{x+y} = 27$, Find x

Solution:

Substitute $y = 2x$ into $3^{x+y} = 27$

Giving us $3^{x+2x} = 3^3$,

equating exponents $\rightarrow x + 2x = 3$

$\therefore x = 1$

Example 3

Solve for x and y given that$3^{(x+1)} = 27$ *and* $4^{(y-2)} = 16$

Solution:

$3^{(x+1)} = 3^3$

$x + 1 = 3$

$x = 3 - 1$

$x = 2$

$4^{(y-2)} = 16$

$2^{2(y-2)} = 2^4$

$2(y - 2) = 4$

$2y - 4 = 4$

$2y = 8$

$y = 4$

Therefore $x = 2, y = 4$

Example 4

Solve the simultaneous equations

$9^{2a+b} = \frac{1}{729}$

$3^a(9^b) = 27$

Solution:

$9^{2a+b} = \frac{1}{729}$

$3^{2(2a+b)} = \frac{1}{3^6}$

$3^{4a+2b} = 3^{-6}$

$4a + 2b = -6$ ……………..$(eqn1)$

$3^a(9^b) = 27$

$3^a(3^{2(b)}) = 3^3$

$3^a \times 3^{2b} = 3^3$

$3^{a+2b} = 3^3$

$a + 2b = 3$ ………………….$(eqn2)$

From $eqn2$, $a = 3 - 2b$

Substituting $a = 3 - 2b$ into $(eqn1)$

$\Rightarrow 4(3 - 2b) + 2b = -6$

$12 - 8b + 2b = -6$

$-6b = -18$ dividing both sides by -6

$b = 3$

From $a = 3 - 2b$

From $a = 3 - 2(3) = 3 - 6 = -3$

Thus, $a = -3$, $b = 3$

Example 5

A bacteria culture doubles every hour. If there are initially 100 bacteria, how many bacteria will be present after 5 hours?

Solution:

Since the culture doubles, the growth factor is 2 (each bacteria becomes 2 after an hour).

We know the initial quantity (100) and want to find the final quantity (let it be Q) after 5 hours (represented by exponent t). The general formula for exponential growth is:

$Q = A\ (growth\ factor)^t$

$Q = 100 \times (2)^6$

$Q = 100 * 32 = 3200$

Therefore, there will be 3200 bacteria after 5 hours.

Example 6

Suppose that a culture initially contains 1000 bacteria and that this number doubles each hour. Write a general formula for the number of bacteria N present after *t* hours

Solution:
After one hour, there are 1000×2 bacteria

After two hours, there are $1000 \times 2 \times 2 = 1000 \times 2^2$ bacteria

After three hours, there are $1000 \times 2^2 \times 2 = 1000 \times 2^3$ bacteria and so on.

Following the pattern, if there are bacteria after t hours, then

$N = 1000 \times 2^t$ bacteria

Theme or Focal Area: Definition and relationship between indices and logarithms and logarithmic equations

Definition: If $N = a^x$, then $\log_a(N) = x$

Relationship between indices and logarithms to solve problems

a. To find the logarithm of a number a to the base b, that is, $\log_b(a)$, we ask the question, 'What power do I raise b to, to obtain a?

b. Taking a logarithm is the inverse process of taking a power. Generally, if $a > 0$ and $x > 0$;

a. $a^{\log_a x} = x$

b. $\log_a a^x = x$

The Logarithm Laws

a. $\log_a(a) = 1$

b. $\log(p^n) = n\log(P)$

c. $\log(NM) = \log(N) + \log(M)$

d. $\log\left(\frac{N}{M}\right) = \log(N) - \log(M)$

Where M, N and P are real numbers

Example 1
Solve the equation: $\log_5(125) = x$

Solution:

The logarithm (log) to base 5 of 125 $(log_5(125))$ means what is the exponent to which we have to raise 5 to get 125. We know 5 raised to the power 3 (5^3) equals 125.

Therefore, $\log_5(125) = 3$, $x = 3$

Example 2
Evaluate $\log_8 4$

Solution:
By writing, $\log_8 4 = x$; we have $8^x = 4$ which gives $2^{3x} = 2^2$;

Equating indices, $3x = 2$, so $x = \frac{2}{3}$

Example 3

Solve for x in the equation $2^x = 7$

Solution:

By taking the logarithms to the base 10 of both sides and use the logarithm laws.

$2^x = 7$

$\log_{10} 2^x = \log_{10} 7$

$x\log_{10} 2 = \log_{10} 7$

$x = \frac{\log_{10} 7}{\log_{10} 2} = 2.807$

Example 4

Change $log_2(7)$ to base 10 (common logarithms)

Solution:

We can change the base of any logarithm using the following rule: $log_a\left(b\right) = \frac{\left(lo\ g_c\left(b\right)\right)}{\left(lo\ g_c\left(a\right)\right)}$

where a and b are any positive numbers and $a \neq 1$.

In this case, we want to change $log_2(7)$ to base 10. So, $log_{10}\left(7\right) = \frac{\left(\log_{10}(7)\right)}{\left(\log_{10}(2)\right)}$

We cannot directly calculate $log_2(7)$ *or* $log_2(10)$ without a calculator.

In practice, you would use a calculator with a log function and approximate the answer.

Example 5

Use logarithmic equation to solve for x given that $3^x = 81$

Solution:

We can convert the exponential equation to a logarithmic equation using the following rule:

$a^x = b$ which is equivalent to $lo\ g_a\left(b\right) = x$

(where a and b are positive numbers and $a \neq 1$)

$log_3(81) = x.$

$\log_3(3^4) = x$

$4\log_3 3 = x$

$4 = x$

Learning Tasks

Guide learners to perform the following task to check for understanding. Support system should be offered to learners who will struggle.

Learners to:

a. review and recollect the initial laws of indices and establish other laws for negative powers and roots

b. recognise the relationship between surds and indices and apply laws of indices to simplify expressions
c. solve simple equations involving indices
d. establish the relationship between indices and logarithms
e. use the properties of logarithms to solve related problems in one base

Pedagogical Exemplars

The objective of the lessons for the week is for all learners to demonstrate knowledge and understanding of numbers about Indices and Logarithms and their real-world applications. Take into consideration the following proposed pedagogical strategies in the curriculum.

Collaborative Learning

Learners will be working in convenient groups (ability, mixed ability, mixed gender, or pairs etc.) to:

a. solve simple indicial equations using problem-based and talk-for-learning approaches
b. apply the exponential equation to model phenomenon involving growths and decays
c. explore the relationship between indices and logarithms
d. solve related problems in one base

Experiential Learning: Learners will collaboratively be engaged in hands-on activity (learning by doing) to

a. explore the laws of indices
b. explore indices with negative exponents
c. apply the exponential equation to real-life problems of growth, decay and compound interests

Key Assessment

Assessment Level 1: Recall

1. Simplify and write the answer with positive indices: $\frac{(x^3)^4}{(x^5)^2}$
2. Simplify and write the answer with positive indices: $\frac{(a^2\, b^4)}{(a^5\, b^3)}$
3. Simplify $2^2 \times 4^{-4} \div 16^{-3}$
4. Find the value of x in the equation $2^3 \times 3^4 \times 72 = 6^x$

Assessment Level 2: Skills of Conceptual Understanding:

a. If $2^n = 32$ find the value of n
b. Solve $3^{3-x} = 27^{x-1.}$
c. Show that
   i. $32^{-\frac{2}{5}} = \frac{1}{4}$
   ii. $\left(2x^{-\frac{2}{5}}\right)^5 = \frac{32}{x^2}$

4. Find the value of x given
   i. $6\,25^{0.17} \times 625^{0.08} = 25^x \times 25^{-\frac{3}{2}}$

5. If $\left(\frac{3}{5}\right)^x = \left(\frac{81}{625}\right)$, then what is the value of x^x

6. Given that $\left(\frac{7}{5}\right)^{4x} \times \left(\frac{7}{5}\right)^{3x-1} = \left(\frac{7}{5}\right)^8$, find the value of x that satisfies this equation

7. Find the value of a if: $5^{3a-1} \times 125 = 2\ 5^{2a-1}$

Assessment Level 3: STRATEGIC THINKING

a. Given that $y = 5x$ and $3^{x+y} = 81$, Find x

b. Solve the simultaneous equations $9^{2a+b} = 2187$ and $3^a \times 9^b = 27$

c. Simplify $8^{\frac{2}{3}}$

d. Simplify $\log_b x^2 + \log_b x^3 - \log_b x^4$

e. Calculate $\log_7 8$ to four decimal places

f. If $\sqrt{5^x} = 25$, find the value of x

g. It is given that $x = \sqrt{3}$ and $y = \sqrt{12}$. Find in the simplest form, the value of

a. xy

b. $\frac{y}{x}$

c. $(x + y)^2$

h. Given $\log_7 2 = \alpha$, $\log_7 3 = \beta$ and $\log_7 5 = \gamma$, express in terms of α, β, and γ ;

a) $\log_7 6$

b) $\log_7 \frac{15}{2}$

Assessment Level 4: Extended Thinking

1. For the rule $y = 20 \times 3^t$

2. Complete the table of values.

t	0	1	2	3
y				

3. Plot the graph of y against t.

4. Find the value y, correct to 2 decimal places, when:

a. $t = 0.5$

b. $t = 2.5$

c. $t = 2$

5. Suppose that GH¢ 5,000 is invested at 6% iinterest compounded annually. In t years an investment will grow to the amount expressed by the function

$$S(t) = 5000 \cdot 1.06^t$$

where t is time (in years).

a. Explain how you will use the idea of logarithms to calculate the number of years it will take to double the initial investment

b. How long will it take to accumulate GH¢ 15,000 in the account?

c. An amount of *Gh¢*100,000 is invested at an annual compound interest rate of 6.25%.

d. If the interest is calculated yearly, how long before the investment has reached *Gh¢*150,000?
a. What is the answer if the interest is paid monthly?
e. On January 1, 2020, Tahiru invested *Gh¢*50,000.00 at 8% per annum compound interest. Interest is only paid on January 1 of each year. After how many years will the investment be worth:
   i. Gh¢75,000
   ii. Gh¢100,000?

Section 2 Review

This section explored the world of indices and logarithms, which has equipped you with tools to tackle a variety of problems within the Modelling with Algebra Strand.

Week 4: Surds

a. We began by delving into **surds**, numbers expressed as radicals (roots). We explored their properties and techniques for simplifying them.
b. A key concept introduced was **rationalisation of surds**, a process of eliminating radicals from the denominator of an expression for easier manipulation.

Week 5: Indices and Logarithms

a. Week 5 expanded our toolkit with **indices** (exponents); a concise way to represent repeated multiplication. We honed our skills in simplifying expressions involving both surds and indices.
b. We tackled **indicial equations**, equations where the variable is present within the exponent.
c. The exciting introduction of **logarithms** occurred, functions that are the inverse of exponentiation. We explored the relationship between exponents and logarithms and their applications in solving **logarithmic equations**.

By mastering these concepts, you can now:

a. Simplify expressions containing surds and indices.
b. Solve equations involving exponents and logarithms.
c. Gain a deeper understanding of the relationship between these powerful mathematical tools.

This newfound knowledge empowers you to approach various problems in science, engineering, and finance, where exponential growth, decay, and logarithmic relationships are frequently encountered.

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SECTION 3: SEQUENCES AND FUNCTIONS

Strand: **Modelling with Algebra**

Sub-Strand: Applications of Algebra

Content Standard: *Demonstrate knowledge and understanding of the application of algebraic processes and reasoning involving sequence, functions, and linear programming*

Learning Outcome(s):

1. *Examine, analyse, determine and predict other terms in a pattern/sequence*
2. *Distinguish among various types of relations, find the domain and range, and evaluate functions*
3. *Show that a function is injective (into) and/or surjective (onto), find the inverse and describe the relationship between two variables and establish composite functions*
4. *Graph linear and quadratic functions and determine the intercepts*
5. *Find graphically and algebraically solutions to a system of three linear equations in three variables and apply them to solve real life problems*
6. *Perform algebraic manipulations on polynomial functions and graph polynomial functions*
7. *Find domain, range, zero of a rational function and state when it is undefined.*

INTRODUCTION AND SECTION SUMMARY

Mathematics gives us a strong language to define patterns, relationships, and variations. This section addresses several basic concepts that form the building blocks of mathematics: We explore ordered lists of numbers, their progressions, and the concept of adding their terms (series) to understand accumulating patterns. We will also delve into connections between sets of numbers, where relations describe any association, and functions establish a specific, clearly defined rule that assigns one output to each valid input. Then we will visualize these functions using straight lines and smooth curves. Linear functions represent proportional change, while quadratic functions model relationships with a turning point, often used in describing real-world phenomena. We will focus on these functions in detail, exploring their properties and how they influence the shape of their corresponding graphs (parabolas). We will introduce functions formed by dividing polynomials, exploring their unique characteristics, behaviour, and how they differ from other function types. By mastering these concepts, you should gain a deeper understanding of how mathematics helps us analyse patterns, solve problems, and model various scenes in the real world.

The weeks covered by the section are:

Week 6:

1. *Definition of a sequence*
2. *Number patterns*
3. *Rules for finding terms of sequences*
4. *Nth term of a pattern*
5. *Types of sequence*

6. *Applications of sequence*

Week 7:

1. *Definition of Relations*
2. *Types of relations*
3. *Evaluation of functions*
4. *Functions*
5. *Types of functions*
6. *Attributes of functions*
7. *Composite functions*

Week 8:

1. *Graphs of linear functions*
2. *Graphs of quadratic functions*
3. *Intercepts of linear and quadratic graphs*

Week 9:

1. *Graphs of linear functions*
2. *Area enclosed by graphs*
3. *Systems of linear equations*
4. *Word problems involving systems of linear equations*

Week 10:

1. *Quadratic functions*
2. *Roots of quadratic equations*
3. *Factor and remainder theorems of polynomial functions*
4. *Graphs of polynomial functions*
5. *Application in real life*

Week 11:

1. *Definition of rational functions*
2. *Domain and zeroes of rational functions*
3. *Decomposition of rational functions into partial fractions*

SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in practical activities on sequences and series, relations and functions, graphs of linear and quadratic functions, quadratic functions as well as rational functions. Learners should be given the platform to work in groups to develop their own real-life questions and find answers. Therefore, Experiential learning activities and Mixed-ability groupings should dominate the lessons on these concepts. All learners, irrespective of their learning abilities should be assisted to take part fully in investigations and presentation of findings. However, make considerations and accommodations for the different groups. That is, offer approaching proficiency learners the opportunity to make oral presentations. Then, extend activities for the above average/highly proficient learners to use formulae and computer applications to solve problems.

ASSESSMENT SUMMARY

Assessment methods ranging from quizzes, tests, and homework assignments can be used to evaluate learners understanding of concepts and their ability to solve problems. Performance tasks like solving real-world problems involving sequences and series, relations and functions, graphs of linear and quadratic functions, quadratic functions and rational functions will also be used to assess learner's application of these mathematical skills. Also, relations and functions will make use of various visual aids such as diagrams of types of mappings; and charts on graphs and interactive mapping tools will also be incorporated to engage learners in hands-on learning experiences. Assessment strategies which vary from Level 1 to Level 4 questions of the DoK will be used. Teacher should record the performances of learners for continuous assessment records.

Week 6

Learning Indicator(s):

1. *Recognize sequences in mathematics as an enumerated collection of objects in which repetitions are allowed and classify sequences into linear or exponential*
2. *Find the nth term of linear and exponential sequences*

Theme or Focal Area(s): Definition of sequences

Patterns occur all around us. The male honeybee hatches from an unfertilized egg, while the female hatches from a fertilized one. The "family tree" of a male honeybee can be represented by ***Figure 1:* where M represents male and F represents female**.

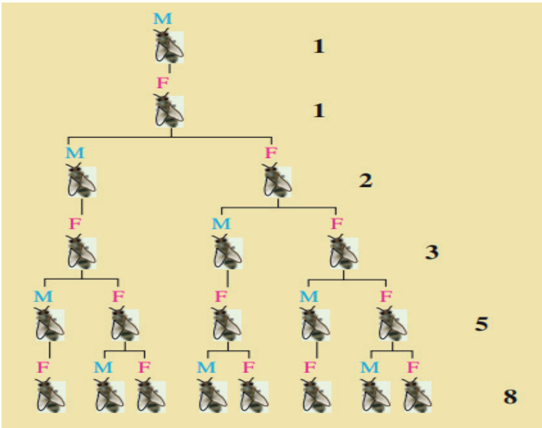


Figure 1: *"Family tree of a male honeybee" (Lial, Hornsby, McGinnis, 2012)*

Starting with the male honeybee at the top, and counting the number of bees in each generation, we obtain the following numbers in the order: 1, 1, 2, 3, 5, 8

Notice the pattern: After the first two terms (1 and 1), each successive term is obtained by adding the two previous terms. This sequence of numbers is called the *Fibonacci sequence*.

A Sequence is any set of objects, often numbers, that follow a particular pattern infinitely. Whilst a series refers to the description of the operation that would add all of the items in a sequence. The objects that make a sequence are called terms and the *nth* term can be represented by S_n, U_n, a_n or similar notations where n is a positive integer indicating the position of the term. For example, U_1 represents the first term and a_4, represents the fourth term in a different sequence.

Problems in various fields and in our daily lives require the application of sequences and series. For example, in trying to predict the value of an investment after some number of years, taking into account the annual interest rate, say, 23%, and a principal of *GH¢*100.00, the following sequence can model the problem and hence used for the prediction: 100, 100(1.23), 100 $(1.23)^2$, 100 $(1.23)^3$, 100 $(1.23)^4$, …

The terms of the sequence indicate how much the investment is worth at the beginning of each year i.e., at the beginning of the first year, it is only worth the principal (*GH¢* 100.00) and at the beginning of the fourth year (at the end of the third year), it is worth *GH¢* 100 $(1.23)^3$. 00 = *GH¢* 186.09. The three dots (ellipses) indicates that the sequence continues indefinitely.

Knowledge equips us with skills and strategies to find the sum of things going on even up to infinity which are useful in scientific thesis.

The pattern that is observed in a sequence can be used to classify sequence and also predict other terms. In this section arithmetic (linear) and geometric (exponential) sequences will be discussed.

Example 1

Given the sequence: 3, 6, 10, 15, 21 …, determine the next term and describe the pattern

Solution:

The sequence is such that, 3 is added to the first number to obtain the second number, 4 added to the second to obtain the third, 5 added to the third to obtain the fourth and 6 added to the fourth to obtain the fifth term hence, the next term should be 21 + 7 = 28

Mathematically, the sequence can be written thus: $a_n = a_{n-1} + n + 1, n \in \mathbb{Z}, n \geq 2, a_1 = 3$ or

$a_n = a_{n-1} + n + 1, n = 2, 3, 4, \ldots, a_1 = 3$

Example 2

Generate the first six terms of the sequence with the rule: $u_{n+1} = 2\, u_n$ and $u_1 = 3$

Solution:

$u_{n+1} = 2\, u_n, u_1 = 3$

When $n = 1$,

$u_{1+1} = 2\, u_1$

$u_2 = 2(3) = 6$

When $n = 2$,

$u_{2+1} = 2\, u_2$

$u_3 = 2(6) = 12$

When $n = 3$,

$u_{3+1} = 2\, u_3$

$u_4 = 2(12) = 24$

When $n = 4$,

$u_{4+1} = 2\, u_4$

$u_5 = 2(24) = 48$

When $n = 5$,

$u_{5+1} = 2\, u_5$

$u_6 = 2(48) = 96$

$\therefore$ the first six terms are 3, 6, 12, 24, 48 and 96

Theme or Focal Area(s): *nth* term of Arithmetic and Geometric sequences

Definition/Introduction

1. Identifying key terms of the sequence such as the first, the common difference and/or ratios, and the nth-term.
2. Establishing the general rule for arithmetic and geometric sequence by using the first terms and relationship between consecutive terms (simple recursions) and the notations of sequences,
3. Determining arithmetic and geometric sequences
4. Identifying a sequence that is neither arithmetic nor geometric
5. Insert arithmetic and geometric means for given sequence.

Arithmetic / Linear Sequence / Progression

If the terms of a sequence after the first term can be found by adding a constant number to the preceding term, that sequence is classified as an Arithmetic Sequence, arithmetic progression or a linear sequence. It can be deduced then, that the difference between one term and the next term, d is a constant i.e., $d = u_{n+1} - u_n = u_n - u_{n-1}\ \forall\ n = 1, 2, 3, \ldots$

Note:

To find the common difference, we subtract a term from the successive term and as subtraction is not commutative, the reverse is inaccurate

An example of a linear sequence is 1, 4, 7, 10, 13, 16, 19, … where the common difference is $u_3 - u_2 = 7 - 4 = 3$, $u_6 - u_5 = 16 - 13 = 3$ or any such expression.

The terms of an arithmetic progression are

$u_1 = a$, $u_2 = (a + d)$, $u_3 = (a + 2d)$, $u_4 = (a + 3d)$, … equivalent to

$u_1 = (a + (1 - 1)d)$, $u_2 = (a + (2 - 1)d)$, $u_3 = (a + (3 - 1)d)$, $u_4 = (a + (4 - 1)d)$, …

The rule for finding the *nth* term of an arithmetic sequence can thus be written as:

$\boldsymbol{U_n = a + d(n - 1)}$ where $\boldsymbol{a}$ is the first term, $\boldsymbol{d}$ the common difference and $\boldsymbol{n}$ is the position of the term.

Example 1

Write the first five terms of the arithmetic sequence with first term −2 and common difference 3

Solution:

Let $u_n = nth\ term = a + (n - 1)d$ where

$a = first\ term = u_1 = -2$

$d = common\ difference = 3$

$u_2 = -2 + 3 = 1$

$u_3 = -2 + 2(3) = 4$

$u_4 = -2 + 3(3) = 7$

$u_5 = -2 + 4(3) = 10$

∴ the first five terms are −2, 1, 4, 7 and 10

Example 2

Find the 29th term is of the arithmetic sequence whose first three terms are −12, −6, 0, …

Solution:

−12, −6, 0, …

$First\ term = -12,$

$Common\ difference = 0 - (-6) = 6$

$29th\ term = -12 + 28(6) = 156$

Geometric / Exponential Sequence

Geometric sequences are sequences in which each term after the first is found by multiplying the preceding term by a nonzero constant. While arithmetic sequences are marked by their common differences, geometric sequences are attributed by common ratios. For example, the terms (except the first) of 3, 6, 12, 24, 48, 96, ... are obtained by multiplying the preceding terms by 2. The common ratio, usually represented by r, is therefore 2. The sequence can hence be rewritten as 3, 3(2), $3\,(2)^2$, $3\,(2)^3$, $3\,(2)^4$, $3\,(2)^5$, …

By observation, the rule for finding the terms of the sequence is $3\,(2)^{n-1}$, where 3 is the first term and 2 is the common ratio. For all geometric sequences,

The nth term, $U_n = a\,r^{n-1}$ where ***a*** is the first term, ***r,*** the common ratio and ***n*** is the number of terms in the sequence. Note that $r = \frac{u_n}{u_{n-1}} = \frac{u_{n+1}}{u_n}$

Example 1

Determine the common ratio for the geometric sequence: $\frac{1}{4}, -1, 4, -16, 64, \ldots$

Solution:

$\frac{1}{4}, -1, 4, -16, 64, \ldots$

$Common\ ratio = \frac{64}{-16} = -4$

Example 2

Find the *nth* term of

a. $5, -10, 20, -40, 80\ldots$

b. $4, \frac{8}{3}, \frac{16}{9}, \ldots$

Solution:

a. $5, -10, 20, -40, 80\ldots$

First term, $a = 5$

Common ratio, $r = -\frac{40}{20} = -2$

nth term, $u_n = a\,r^{n-1}$

$= 5\,(-2)^{n-1}$

b. $4, \frac{8}{3}, \frac{16}{9}, \ldots$

First term, $a = 4$

Common ratio, $r = \frac{8}{3} \div 4 = \frac{2}{3}$

nth term, $u_n = a\,r^{n-1}$

$= 4\left(\frac{2}{3}\right)^{n-1}$

Example 3

The fifth and ninth terms of an exponential sequence are 16 and 256 respectively. Find

a. the first term, a and the common ratio, r if $r > 0$

b. the 11th term

Solution:

a. *Fifth term*, $u_5 = a\,r^4 = 16$

Ninth term, $u_9 = a\,r^8 = 256$

$$\left.\begin{aligned} a\,r^4 &= 16 \\ a\,r^8 &= 256 \end{aligned}\right\}$$

$$\frac{a\,r^8}{a\,r^4} = \frac{256}{16}$$

$r^4 = 16$

$r = \sqrt[4]{16} = 2$

$a(2^4) = 16$

$16a = 16$

$a = 1$

b. 11*th term*, $u_{11} = a\,r^{10}$

$= 1(2)^{10}$

$= 1024$

Learning Tasks

Guide learners to solve the following task and check for understanding. Support system should be provided to learners who will struggle.

Guide learners to:

a. *find terms of arithmetic and geometric sequences*

b. *find terms of sequences which may not be arithmetic nor geometric*

c. *find the Nth term of a pattern and solve real-world situations involving sequences.*

d. *find terms of sequences that require solutions to systems of equations and solving real-life problems involving sequences*

Pedagogical Exemplars

The aim of the lessons for the week is for all learners to be able to recognise and find rules governing sequences and number patterns, find the Nth term of a given sequence, recognise the types of sequence given a pattern and solve real-life application problems of sequences. The following pedagogical approaches are suggested for facilitators to take learners through.

1. **Project-Based Learning**

   Learners will collaboratively embark on a project to

   a. come up with sequence that is relevant and applicable to real life and make presentations

   b. undertake a research inquiry into what happens when terms of sequence are summed up

   c. model sequences (numerical and non-numerical - using of colours, cut-out shapes, objects, fabrics, sound, actions, etc.) and talk about the pattern rule.

2. **Collaborative Learning**

   In convenient groups (ability, mixed ability, mixed gender, or pairs etc.), learners

   a. create sequences using objects, symbols and numerals and discuss pattern rules
   b. investigate, identify and discuss real-life patterns in their environment and talk about the pattern rule. For instance, have learners act out and talk about:
   c. Stacking cups, chairs, bowls etc., to compare the number of objects to the height of the object
   d. Seating around tables: Think about a restaurant, where a square table fits 4 people. What happens when two square tables are put together, now 6 people are seated. Put 3 square tables together and now 8 people are seated etc. Learner may visualise this better if they are allowed to model this sequence
   e. Filling something: Think about the rate at which the object is being filled versus time would be the variable. An example could be a sink being filled or a pool being filled. Also, consider draining a liquid or sand or sugar etc., out of a container having a small hole beneath
   f. Fencing and perimeter: Discuss how adding a fence panel to each side of a rectangular fence would change the perimeter. For example, the first figure (fence) could have one panel on each side (or change it so it is not a square). The next figure could have two panels on each side and so on. Each time, find the new perimeter
   g. Negative number patterns: Think about temperature and places below sea level on earth. Create situations like, during a rainfall the surface of the water started at 153 feet below sea level and rose at a rate of such and such per hour or diving in the ocean.

3. **Experiential Learning**

   a. Learners (in pairs or groups) work with sequence of numbers to establish the general rule for arithmetic and geometric sequence (by using the first terms and relationship between consecutive terms (simple recursions)) and the notations of sequences, and determine if a sequence is arithmetic or geometric and identify key terms of the sequence such as the first, the common difference and/or ratios, and the nth-term

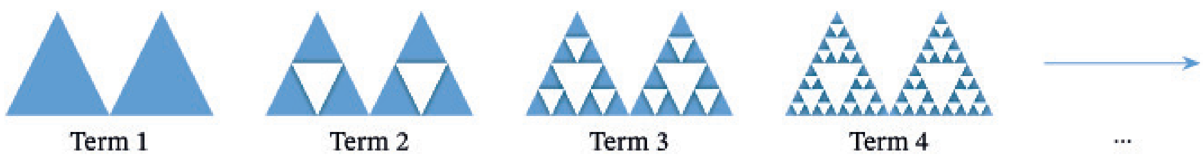
4. **Enquiry based learning**

   a. Learners use research resources (textbooks, electronic devices and any additional relevant resources) to discover applications of sequences in real life

Key Assessment

Assessment Level 1: Recall

Consider the following pattern.



1. Which of the following sequences represents the number of solid blue triangles in each successive term of the pattern?
   a. 2,8, 26,80, …
   b. 1,3, 9,27, …
   c. 2,6, 18,54, …
   d. 2,4, 12,36, …
   e. 2,4, 8,16, …

2. Which type of sequence is found when counting the number of solid blue triangles in the above pattern?

3. Consider the sequence: 1, 4, 27, 256, … What is the next term, and what is the pattern governing this sequence?

Assessment Level 2: Skills and conceptual understanding

1. What kind of sequence is the following: $1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$?

a. Neither geometric nor arithmetic
b. Geometric and arithmetic
c. Geometric only
d. Arithmetic only

2. Which of the following sequences is not classified as arithmetic or geometric?

a. $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$
b. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
c. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 1\frac{}{6}, \ldots$
d. $\frac{1}{3}, \frac{-1}{3}, -1, \frac{-5}{3}$
e. $1, \frac{1}{2}, 0, \frac{-1}{2}, \ldots$

3. Find the 10*th* term of the arithmetic sequence: 2, 5, 8, 11, ...

4. Given the arithmetic sequence: 3, 7, 11, 15, ... Find an expression for the nth term of this sequence.

5. For the geometric sequence: 4, 12, 36, 108, ... Find the formula for the nth term of the sequence.

Assessment Level 3: Strategic Reasoning

a. Complete the table for the sequence given by the formula $u_n = 13\ (2)^{n-1}$; $n > 0$

n	1	2	3	4	5	6
$U_n = 13\ (2)^{n-1}$						

Assessment Level 4: Extended thinking

b. Study the table below, write a general rule for y in terms of x and complete the table.

x	1	2	3	4	5	6
y	0	4		24	40	

c. The set of whole numbers is partitioned into subsets with the first number in the first subset, the next two numbers in the second subset, and the next three numbers in the third subset and so on. Find in terms of n a formula for the first member of the *nth* subset.

NB: Start with number of whole number(s) in the first partition, second partition, etc.

Week 7

Learning Indicator(s):

1. *Identify and describe the relationship of patterns, recognise the difference between a relation and a function and write a function that describes a relationship between two quantities in real life situations*
2. *Use function notation to evaluate functions for inputs in their domain and outputs in the co-domain*
3. *Determine the composite of two given functions*
4. *Establish, describe and determine bijective functions and composite function*
5. *Find the inverse of simple functions including one-to-one functions*
6. *Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse)*

Theme or Focal Area(s): Definition of Functions

In week 6, we discussed sequences and series. In that lesson, we looked at the relationship between the position of a term, represented by a variable, n and the term, also represented by another variable, $u\ n$ or $a\ n$. The rule or formula for finding specific terms of a sequence, shows this relationship and hence is a function. The concept of functions has gone through a lot of development and notable among the mathematicians who have worked on its development are René Descartes, Wilhelm Leibniz, Leonhard Euler, Nicolas Bourbáki and John Tate etc. Functions are the building blocks for modelling real life scenarios and thereby, designing machines, predicting natural disasters, curing diseases, understanding world economies and for keeping airplanes in the air etc.

A function, f from set A to set B could be defined as;

a. a pairing of elements in A with elements in B in such a way that each element in A is paired with *exactly one* element in B or

b. a rule or relation between A and B that assigns each element $a \in A$ to a *unique* element, $b \in B$.

From the definitions of functions, it can be inferred that "***All functions are relations BUT not all relations are functions***". A function must assign only ONE output to each input. One-to-one and many-to-one relations are therefore the two types of relations that qualify to be functions.

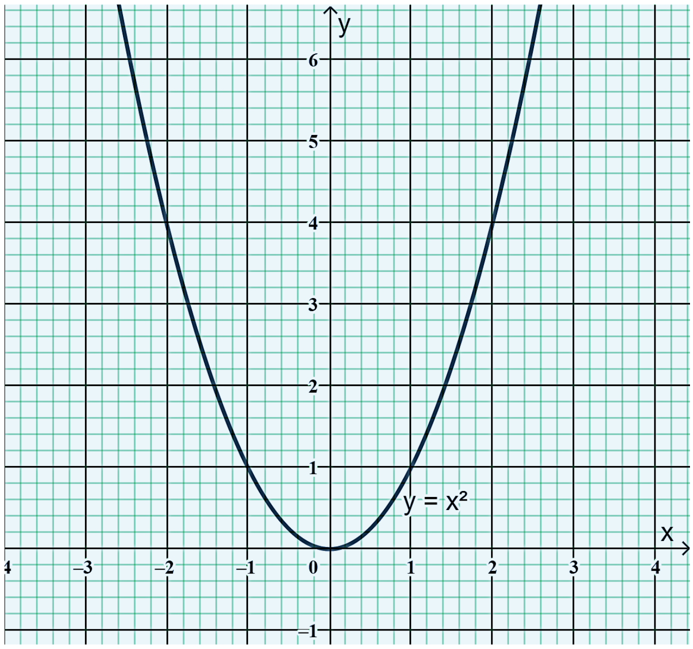
If a function describes the relationship between a number and its square, it can be expressed as

a. $\{(-4, 16), (-3, 9), (-2, 4), (-1, 1), (0, 0), (0.5, 0.25), (1,1), (2, 4)\}$ i.e., as ordered pairs,

b. $f(x) = x^2$ read as "*f of x*" or "*f* at *x*" i.e., using a formula,

c. $f{:}x \to x^2$ read as "*f is such that x maps on to* x^2 i.e., using function notation or

d. graphically, as shown below



The vertical line test

This is a test that checks whether or not any vertical line drawn on the same graph as the curve of an equation cuts the curve at more than one point. If the vertical line cuts the curve at more than one point, the curve is said to have failed the test and vice versa. The graph of a function must pass the vertical line test.

Considering Figure 2, in illustration 1, the vertical line cuts the curve at two points, *A* and *B* and thus, the curve does not pass the vertical line test. Automatically, it does not qualify to be the graph of a function. Actually, the curve is the graph of $y^2 = x$. Considering that x represents elements in the domain and y elements in the co-domain, the rule, $y^2 = x$ represents a one-to-many relation and confirms the conclusion that $y^2 = x$ is not a function.

The curve in illustration 2 however, passes the vertical line test as the vertical line cuts the curve at only one point i.e., *A* and hence the equation / rule represented by the curve is a function.

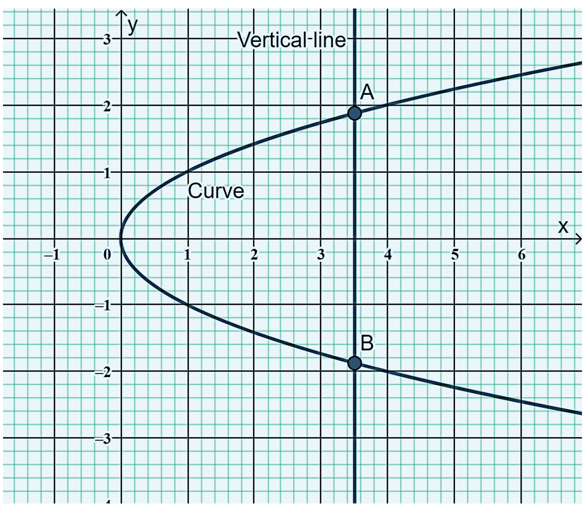


Illustration 1

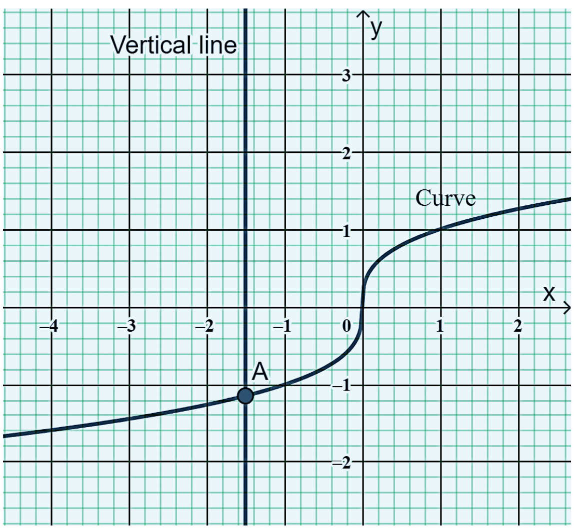


***Illustration* 2**

Figure 2: *Vertical line test*

Theme or Focal Area(s): Evaluating functions

For a function, f defined as $f(x) = x^2$, $f(x)$ is the functional value of x, it is the image of x and on a two-dimensional plane, it is the corresponding value of x for the point which lies on the graph of x^2. For example, given that $g(x) = -2x$, $g(-3)$ means, apply the rule "*negative of a double of a number*" to −2 and hence, $g(-3) = -2(-3) = 6$

Example 1

If $r(x) = \frac{3}{7x - 14}$, evaluate $r(3)$

Solution:

$r(x) = \frac{3}{7x - 14}$

$r(3) = \frac{3}{7(3) - 14} = \frac{3}{7}$

Example 2

Given $h : x \rightarrow 5x - \frac{3}{x}$, find the image of 2 under h

Solution:

$h(x) = 5x - \frac{3}{x}$

$h(2) = 5(2) - \frac{3}{2}$

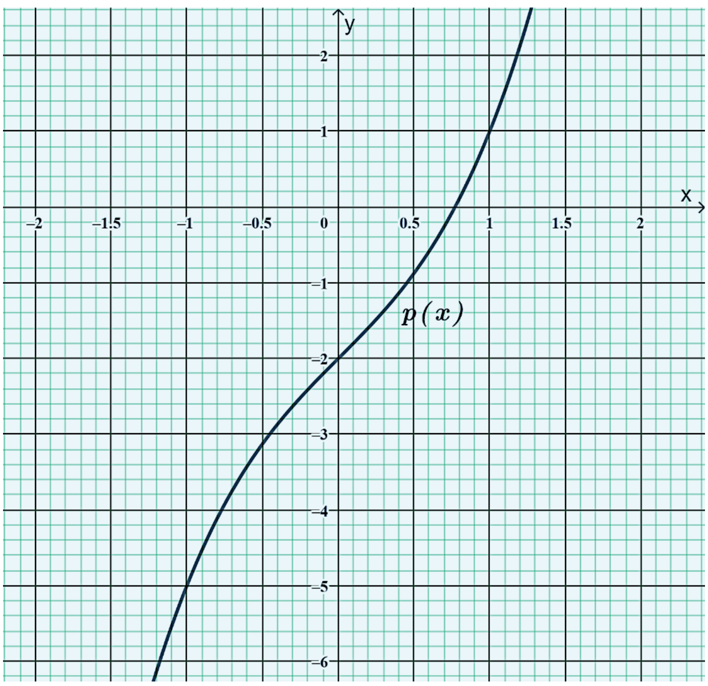
$= 10 - \frac{3}{2}$

$= 8.5$

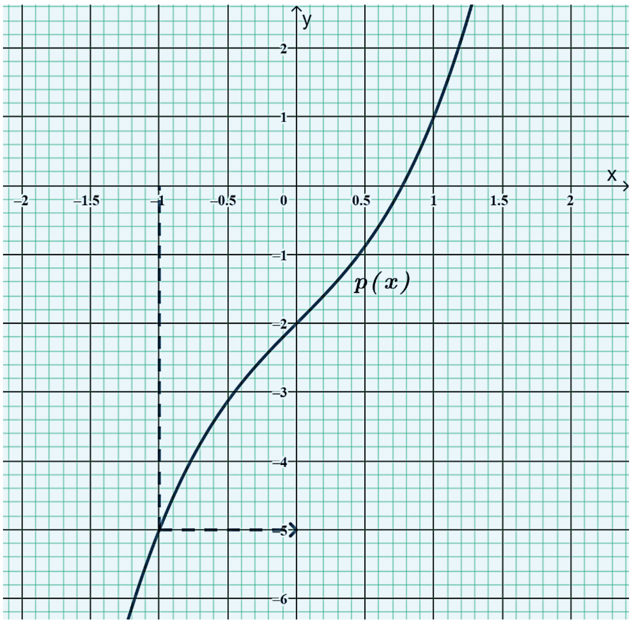
Hence the image of 2 under $h : x \rightarrow 5x - \frac{3}{x}$ is 8.5

Example 3

The figure below shows the path, $p(x)$ taken by a particle in the Cartesian plane. Use the graph to predict the vertical distance of the particle from the *x-axis* when it is 1 unit to the left of the *y-axis*



Solution:



From the graph, when $x = -1$, $y = -5$ and thus, the vertical distance of the particle from the $x - axis$ when it is 1 unit to the left of the $y - axis$ is 5

Example 4

Temperature conversion is a common task in everyday life, especially when dealing with different temperature scales. Two popular temperature scales are Celsius (°C) and Fahrenheit (°F). To convert temperatures between these scales, we can use a function that takes an input on one scale and produces an output in the other scale.

The function C to F, denoted by $F(C)$ can be used to convert temperatures from Celsius (°C) to Fahrenheit $(°F)$. It can be represented as: $F\left(°F\right) = \frac{9}{5} * C\left(°C\right) + 32$

In this function, C(°C) represents the input temperature in Celsius, and F(°F) represents the output temperature in Fahrenheit.

For example, $25°C$ in degree Fahrenheit, given by $F(°F) = \frac{9}{5} * 25 + 32 = 77°F$

Addition and multiplication properties of a function

- **a.** If f_1 and f_2 are two functions from A to B, then $f_1(x) + f_2(x)$ is defined as-:$(f_1 + f_2)x = f_1(x) + f_2(x)$.
- **b.** If f_1 and f_2 are any two functions from A to B, then $\left(f_1 \cdot f_2\right)x$ is defined as: $(f_1 \times f_2)x = f_1(x) \times f_2(x)$.
- **c.** Function Equality: Two functions are equal only when they have same domain, same co-domain and same mapping elements from domain to co-domain

Example

Given that $f(x) = 3\,x^3 - 2x$ and $g(x) = -2x + 4$, evaluate

- **a.** $(f + g)(-2)$
- **b.** $(f \times g)(5)$

Solution:

$f(x) = 3x^3 - 2x$ and $g(x) = -2x + 4$

a. $(f+g)(x) = 3x^3 - 2x + (-2x + 4)$

$= 3x^3 - 2x - 2x + 4$

$= 3x^3 - 4x + 4$

$(f+g)(-2) = 3(-2)^3 - 4(-2) + 4$

$= -12$

b. $(f \times g)(x) = (3x^3 - 2x)(-2x + 4)$

$= (-6)x^4 + 12x^3 + 4x^2 - 8x$

$(f \times g)(5) = (-6)(5)^4 + 12(5)^3 + 4(5)^2 - 8(5)$

$= -2190$

Theme or Focal Area(s): Domain, Range and zeroes of functions

The domain of a function, say $f(x)$ is the set of all values of x which can be evaluated and hence its image under the function can be obtained. For example, if for $f(x)$, $f(a)$, (where a represents any element in the set, D) can be evaluated, then the domain of $f(x)$ is D.

The range of a function is thus the set of all functional values or images of the elements in the domain.

Mathematically,*Domain*, $D = \{a : \exists b \in Y;\ (a, b) \in f\}$ read as *"D is the set of 'a' such that for each 'a', there exists an element, 'b' in set Y, (the range of the function), where* (a, b) *is a pair from the function, f"* and

Range, $Y = \{b : \exists a \in D;\ (a, b) \in f\}$ read as *Y is the set of 'b' such that for each 'b', there exists an element, 'a' in set D, (the domain of the function), where* (a, b) *is a pair from the function, f*

For a polynomial function, $y = f(x)$

a. *Domain* $= \{x : x \in \mathbb{R}\}$ i.e., the set of all real numbers

b. *Range* $= \{y : y \in \mathbb{R}\}$ i.e., the set of all real numbers

For a rational function, $y = R(x) = \dfrac{p(x)}{q(x)}$

c. *Domain* $= \{x : x \in \mathbb{R},\ q(x) \neq 0\}$ i.e., the set of all real numbers except x values for which $q(x) = 0$

d. The range of rational functions is the domain of the corresponding inverse function and hence cannot be written in a general form

For a radical function, $y = f(x) = \sqrt{p(x)}$

e. *Domain* $= \{x : x \in \mathbb{R},\ p(x) \geq 0\}$ i.e., the set of all real numbers except values of x for which $q(x)$ is negative ($q(x) < 0$)

f. The range of radical functions is the domain of the corresponding inverse function and hence cannot be written in a general form

For an exponential function, $y = f(x) = a\,b^x$

g. *Domain* $= \{x : x \in \mathbb{R}\}$ i.e., the set of all real numbers

h. Range:

a. For $b = 1$, the range of $f(x) = a\,b^x$ is simply $\{a\}$.
b. For b other than 1 and $a > 0$, the *range* $= (0, \infty)$.
c. For b other than 1 and $a < 0$, the *range* $= (-\infty, 0)$

For a logarithmic function, $y = f(x) = \log(x)$

i. *Domain* $= \{x : x \in \mathbb{R}, x \geq 0\}$ i.e., the set of all positive real numbers

j. *Range* $= \{y : y \in \mathbb{R}\}$ i.e., the set of all real numbers

Example 1:
The domain of the function $f(x) = \frac{1}{x-2}$ is the set of all real numbers except $x = 2$ i.e., Domain of y is $\{x : x \in \mathbb{R}, x \neq 2\}$

Example 2:
State the largest possible domain of the function defined by $f : x \rightarrow \frac{2x-1}{2x^2-9x-5}$

Solution

$f(x) = \frac{2x-1}{2x^2-9x-5}$

The function, $f(x)$ is a rational function and hence, is only defined if $2x^2 - 9x - 5 \neq 0$

If $2x^2 - 9x - 5 = 0$,

$2x^2 - 10x + x - 5 = 0$

$2x(x-5) + (x-5) = 0$

$(x-5)(2x+1) = 0$

$x - 5 = 0$ or $2x + 1 = 0$

$x = 5$ or $x = -\frac{1}{2}$

$\therefore$ *Domain* $= \left\{x : x \in \mathbb{R}, x \neq -\frac{1}{2}, x \neq 5\right\}$

Example 3
Find the domain of g if $g : x \rightarrow \sqrt{3x-4}$

Solution:

Since $g(x)$ is a radical function, the expression, $3x - 4$ must be positive for the function to be defined

$3x - 4 \geq 0$

$3x \geq 4$

$x \geq \frac{4}{3}$

$\therefore$ *Domain of* $g(x) = \left\{x : x \in \mathbb{R}, x \geq \frac{4}{3}\right\}$

Zeroes of a function

The zero of a function, say $p(x)$ are the values of x for which $p(x) = 0$

Example 1
Find the zeroes of the function, $f : x \rightarrow \frac{x+3}{x-2}$

Solution:

$f(x) = \frac{x+3}{x-2}$

For $f(x) = 0$,

$\frac{x+3}{x-2} = 0$

$x + 3 = 0$

$x = -3$

The zero of f is thus -3

Example 2

State the zeros of the function defined by $f : x \rightarrow \frac{2x-1}{2x^2-9x-5}$

Solution:

For $f(x) = \frac{2x-1}{2x^2-9x-5} = 0$,

$2x - 1 = 0$

$x = \frac{1}{2}$

Theme or Focal Area (s): Surjective and Injective Functions

For two sets, A and B the following statements hold

- **a.** A function $f : A \rightarrow B$ is called surjective (or is said to map A onto B) if $B =$ *range of f*.
- **b.** A function $f : A \rightarrow B$ is called injective (or one-to-one) if, for all a_1 and a_2 in A, $f(a_1) = f(a_2)$ implies that $a_1 = a_2$.
- **c.** The graph of an injective function must pass the horizontal line test
- **d.** A function $f : A \rightarrow B$ is called bijective if it is both surjective and injective.
- **e.** A function f is strictly increasing if $f(x) > f(y)$ when $x > y$.
- **f.** A function f is strictly decreasing if $f(x) < f(y)$ when $x < y$.
- **g.** A function f is increasing if $f(x) \geq f(y)$ when $x > y$.
- **h.** A function f is decreasing if $f(x) \leq f(y)$ when $x < y$.

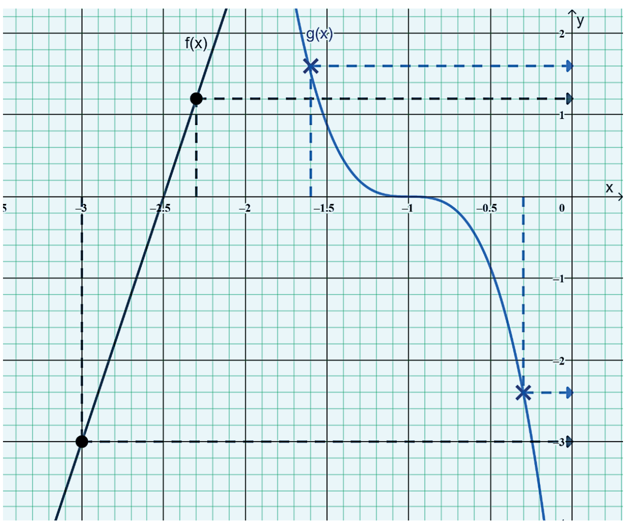


Figure 3: *Increasing and decreasing functions*

From Figure 2, $f(x)$ is a one-to-one function since no part of the graph of $f(x)$ is horizontal (parallel to the $x - axis$). No more than one value of x has the same value of y. Likewise, $g(x)$ is a one-to-one function as for every position on the $x - axis$ (value of x), there is but one height (corresponding value of y). Both curves pass the horizontal line test (any horizontal line drawn will cut the curves at only point each) and the only confirms that they are injective.

Also, $f(x)$ can be described as a strictly increasing function since for $x = -3$, which is less than $x = -2.3$, $y = -3$ which is also less than $y = 1.2$ (corresponding to $x = -2.3$). the reverse can be said of $g(x)$. The height of $g(x)$ for a smaller value of x, i.e., $x = -1.6$ is higher than a bigger value of x, i.e., $x = -0.3$. As the graph of $g(x)$ is traced from left to right (increasing values of x), the graph goes downwards (decreasing values of y) hence $g(x)$ is a strictly decreasing function

Conclusions:

- **a.** A function is one-to-one if it is either strictly increasing or strictly decreasing.
- **b.** A one-to-one function never assigns the same value to two different elements of the domain.
- **c.** For an onto function, range and co-domain are equal.

Example

If $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 3x + 7$, show the function above is one-to-one

Solution:

$f(x) = 3x + 7$

$f(y) = 3y + 7$

For one – to – one function;

$f(x) = f(y) \Rightarrow x = y$

$3x + 7 = 3y + 7$

$3x = 3y$

$x = y$

The function is one – to – one.

Examples

Bijective function example: Let's consider a real-life scenario of matching students to their unique student IDs. Suppose we have a class of students and a set of unique student identification numbers. A bijective function can be established between the set of students and the set of student IDs, ensuring that each student is assigned a distinct student ID, and no two students have the same ID.

Theme or Focal Area(s): Inverse of functions

The inverse of bijection f denoted as f^{-1} is a function which assigns to b, a unique element a such that $f(a) = b$. hence $f^{-1}(b) = a$

The inverse of a function is visually represented as the reflection of original function over the line $y = x$ as shown in Figure 4 . By definition, functions that have inverses (known as invertible functions) must be bijective and inverse functions are also bijective and hence pass the vertical line and the horizontal line tests

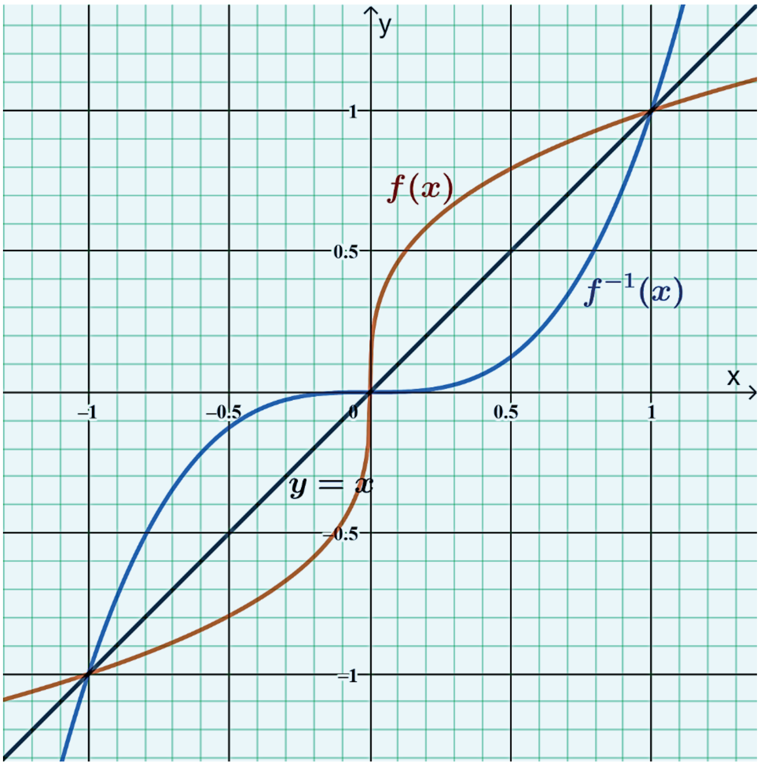


Figure 4: *Inverse functions*

Examples

You are the manager of a car rental company, and you want to determine at what time during the day the number of available rental cars reaches a specific number (let's say 20 cars). You know that the number of available rental cars (C) at any given time of the day (t) follows a simple linear relationship, represented by the function $f(t) = 40 - 2t$. Now, you want to find the time (t) at which the number of available rental cars is exactly 20 ($C = 20$).

Set up the equation:

We are given the function $f(t) = 40 - 2t$, and we want to find t when $f(t) = 20$.

So, the equation becomes:

$f(t) = 20$

$40 - 2t = 20$

Solve for t:

To solve for t, we need to isolate the variable t on one side of the equation. Let's proceed with the steps:

Subtract 40 from both sides of the equation:

$-2t = 20 - 40$

$-2t = -20$

Now, divide both sides by −2 to solve for t:

$$t = \frac{(-20)}{(-2)}$$

$t = 10$

Interpretation:

The solution $t = 10$ tells us that at 10 units of time (which could be hours, minutes, etc., depending on the context), the number of available rental cars will be exactly 20.

Write the expression for the inverse function:

Since the function $f(t) = 40 - 2t$ is a one-to-one function (it represents a straight line with a non-zero slope), it has an inverse function.

The inverse function of $f(t)$ is denoted as $f^{-1}(t)$. To find the inverse, interchange the roles of t and $f(t)$ and solve for $f^{-1}(t)$:

$f(t) = 40 - 2t$

Swap t and $f(t)$:

$t = 40 - 2\,f^{-1}(t)$

Now, solve for $f^{-1}(t)$

$2\,f^{-1}(t) = 40 - t$

Finally, divide by 2:

$f^{-1}(t) = \frac{(40 - t)}{2}$

So, the expression for the inverse function is $f^{-1}(t) = \frac{(40 - t)}{2}$.

Interpretation: The inverse function tells us that given the number of available rental cars (t), we can determine the time $\left(f^{-1}(t)\right)$ at which that specific number of cars will be available.

*(**NB**: Keep in mind that in real-life scenarios, functions and their inverses might not always have straightforward interpretations, and the context of the problem will play a crucial role in understanding their meanings).*

Theme or Focal Area: Composite functions

Consider a real-life scenario where you are running a food delivery service. You have three functions:

a. The first function calculates the total cost of the food order based on the items selected and their prices.

b. The second function calculates the delivery charge based on the distance between the restaurant and the customer's location.

c. The third function calculates the total time it will take to deliver the order, including preparation and delivery time. Now, to find the total cost for a specific food order, you can create a composite function by combining these three functions:

Total cost = Third Function (Second Function (First Function (food_items). In this example, the output of the first function (total cost of food items) becomes the input for the second function (delivery charge calculation), and then the output of the second function becomes the input for the third function (total time calculation).

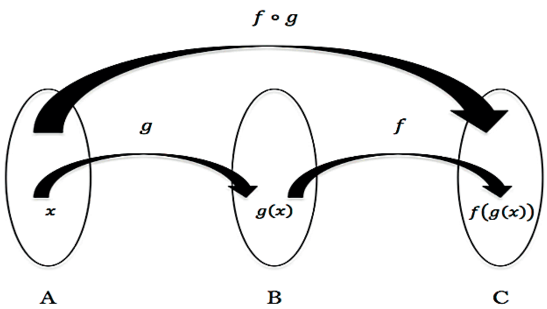


Figure 5: *Composite functions*

Some common properties of composite functions. Properties include:

a. $fog \neq gof$, except $g(x) = f(x)$

b. $(f^{-1} \circ f)\ (a) = f^{-1}(f(a) = f^{-1} f(a) = a$

c. $(f \circ f^{-1})(b) = f^{-1}(f(b) = f^{-1} f(b) = b$

d. $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

e. If f and g both are one to one function, then $f \circ g$ is also one to one.

f. If f and g both are onto function, then $f \circ g$ is also onto.

g. If f and $f \circ g$ both are one to one function, then g is also one to one.

h. If f and fog are onto, then it is not necessary that g is also onto.

Examples

If $h(x) = -2\,x^2 + 3x$ and $f(x) = 3x - 2$, evaluate

a. $(h \circ f)(-2)$

b. $(f \circ h)(-2)$

Examples

$h(x) = -2x^2 + 3x$ and $f(x) = 3x - 2$

a. $(h \circ f)(x) = -2(3x - 2)^2 + 3(3x - 2)$

$= -2(9x^2 - 12x + 4) + 9x - 6$

$= -18x^2 + 33x - 14$

$(h \circ f)(-2) = -18(-2)^2 + 33(-2) - 14$

$= -152$

b. $(f \circ h)(x) = 3(-2x^2 + 3x) - 2$

$= -6x^2 + 9x - 2$

$(f \circ h)(-2) = -6(-2)^2 + 9(-2) - 2$

$= -44$

Learning Tasks

Guide learners to solve the following task to check for their understanding. Support system should be offered to learners who will struggle.

Guide learners to:

a. create different types of functions

b. use mappings diagram and algebraic proofs to show whether or not a given function is one-to-one or onto or both

c. explore increasing and decreasing functions by verifying graphically and algebraically

d. create a graph by hand and by appropriate technology to illustrate a function.

e. determine the domain, intercepts, formula for the function and maximum and minimum values of the graph

Pedagogical Exemplars

This week's lessons aim to help learners identify and describe the relationship of patterns, recognise the difference between a relation and a function and write a function that describes a relationship between two quantities in real life situations. Learners will be guided to use function notation to evaluate functions for inputs in their domain and outputs in the co-domain, determine the composite of two given functions, establish, describe and determine bijective functions and composite function, find the inverse of simple functions including one-to-one functions. Learners will be able to solve equations of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

The following pedagogical approaches are suggested to be administered to facilitate smooth classroom discourse that will help meet the needs of learners at different levels of ability.

1. **Problem-based Learning**

   Learners in their various groups should:

   i. investigate the domain of functions such as: polynomials, radical, rational, logarithmic and exponential with the use of calculators and graphing utilities and then share their findings with the class

ii. undertake an inquiry into domain and range of functions in real life
iii. Learners work in convenient groups (mixed gender, ability mixed ability, or pairs) discuss the domain and range of a function.

2. **Collaborative Learning**

   In convenient groups (ability, mixed ability, mixed gender, or pairs etc.), learners

   i. evaluate functions, determine their domains and ranges, and find the composite of functions
   ii. determine bijective functions and find the inverse of such functions (learners must find the inverse function by using graphic illustrations and through algebraic means)
   iii. debate and discuss the types and features of functions and relations

3. **Exploratory Learning**

   Learners construct graphs of given functions by hand in graph books or by use of Edtech tools such graphic calculators GeoGebra, Demos, PhET Simulations, Geometer's Sketch Pad etc. then identify and interpret key points such as the intercepts, turning points, maximum values, minimum values

4. **Experiential Learning**

   Learners will collaboratively be engaged in hands-on activity (learning by doing) to

   i. create a graph by hand and by appropriate technology to illustrate a function, (creators must know the intercepts, maximum and or minimum values that define the graph). Then, have them swap with a different group/pair. The new pair should determine the domain, intercepts, formula for the function and maximum and minimum values of the graph. Then, they should swap back for creators to assess the accuracy or otherwise with justification)

Key Assessment

Assessment level 1: Recall and reproduction

1. Evaluate $g(-3)$ if $g(x) = 2\,x^2 - 4x + 7$
2. A function has the set of all real numbers as its domain and range. This group of function include linear and quadratic functions. What group of functions does it belong to?
3. *Let* $f\,(x) = 3x + 2$ *and* $g(x) = 2x^2 - 1$. Determine the composite function $(g \circ f)(x)$
4. *Given* $f(x) = \frac{1}{2}$ *and* $g(x) = ln(x)$, find the composite function, $(g \circ f)(x)$.
5. Given the functions $f(x) = 3x$ and $g(x) = x - 2$, find the composite function $(f \circ g)(x)$
6. Given the function $f(x) = 2x + 3$, evaluate $f(4)$
7. Consider the function $h(t) = 3t^3 - 2t^2 + 5$. Calculate $h(-1)$.
8. Consider the function $v(t) = ln(t^2 + 1) - \sqrt{(t)}$. Calculate $v(4)$.
9. State the domain of $P(x) = -3\,x^3 + 6x - 7$
10. State the domain of $f(x) = \sqrt{2x + 3}$,
11. Find the domain of function $g(t) = 3 + x\ \_\ x\ 2 - 4x + 3$

Assessment Level 2: Skills and Conceptual Understanding

1. Find the inverse of the function $f(x) = 2x + 5$
2. Find the inverse of the function $h(x) = 4 - 3x$
3. Find the inverse of the one-to-one function $p(x) = 5x - 7$
4. Determine whether the function $f(x) = 2x + 3$ is bijective

5. For $q(x) = -3\ x^2 + 7$, determine which interval strictly increases and which interval strictly decreases
6. If an astronaut weigh 65 *kg* on the surface of the earth, then her weight when she if *d* miles from the earth is given by

$$w(d) = 65\left(\frac{3960}{3960 + d}\right)^2$$

7. Determine her weight when she is 150 miles above the earth.
8. Construct a table of values for the function *w* that gives her weight at heights in the interval: [0, 300] using 50 miles for the increment.
9. From your table, describe the nature of the function, $w(d)$, (increasing or decreasing, injective, surjective and bijective).
10. Construct the graph of the inverse function of the function whose graph is shown in Figure 6

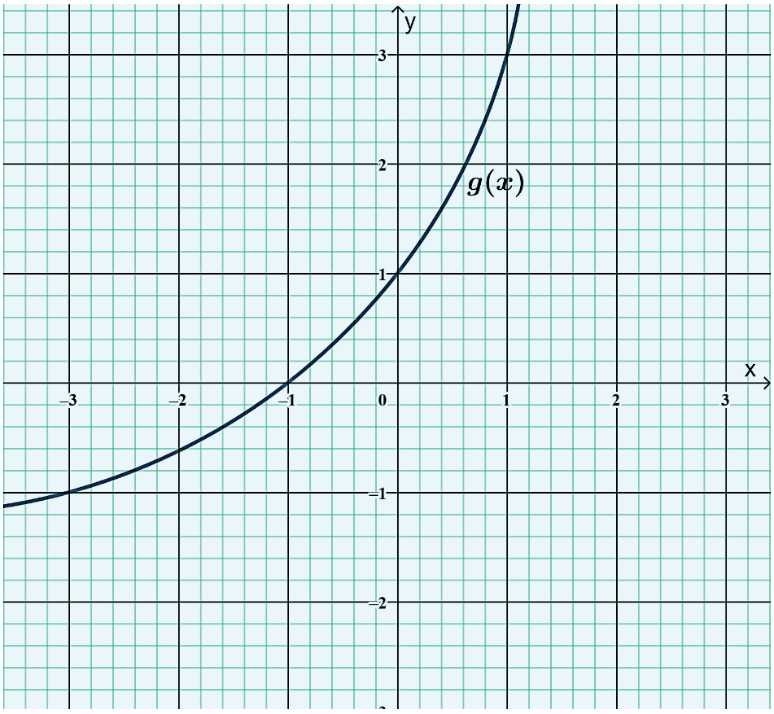


Figure 6

Assessment Level 3: Strategic reasoning:

a. Establish whether the function $g(x) = x^2$ is bijective on the domain of real numbers.
b. Consider the functions $p(x) = x^3$ and $q(x) = \sqrt{x}$. Find the composite function $(q \circ p)(x)$

Assessment Level 4: Extended critical thinking and reasoning:

1. A class teacher decides to use the following rules to organise the seating arrangements in the class:
   i. Ewes sit in the front row
   ii. Christians and traditionalists sit in the second row
   iii. Ashanti and Ga learners sit in the third row and
   iv. Girls sit in the fourth row

If some Ewes are Muslims but none are Christians or traditionalists, all Ashanti learners are girls and most Ga learners are traditionalists, explain with reasons, whether or not the composite function of all the functions (rules) will be bijective.

Week 8

Learning Indicator(s):

1. *Recognise and construct linear and parabolic functions by hand and with the aid of technology, where appropriate show the intercepts and investigate the turning points*
2. *Recognise and use appropriate algebraic notions, properties of linear and non-linear functions, linear and non-linear equations and solve linear and non-linear simultaneous systems*
3. *Recognise linear equations in two variables, draw its graph by hand and by using the appropriate technology (e.g., GeoGebra, Demos, PhET Simulations, Geometer's Sketch Pad), find the solution using the graph and determine the area enclosed by the graphs*
4. *Recognise and model statements into linear mathematical equations and solve simultaneously*

Introduction

Linear and parabolic functions are among the most essential types of functions. They are a part of a wider range of functions called polynomial functions. Whether you are charting the path of a projectile, analysing trends in data, or simply modelling real-world scenarios, these functions play a crucial role.

Linear functions have a constant rate of change, which makes them ideal for representing relationships that evolve uniformly over time or space. They serve as the building blocks for more complex mathematical concepts and are prevalent in various fields, from economics to physics.

Parabolic functions, on the other hand, exhibit a curved, symmetrical shape and are characterized by a squared term. They often describe phenomena such as the trajectory of a thrown object, the shape of a suspension bridge cable, or the arc of a fountain's water.

This week's lessons will focus on the properties, applications, and graphical representations of both linear and parabolic functions. Through exploration, problem-solving, and real-world examples, learners will gain a deeper understanding of these fundamental mathematical concepts and their significance in everyday life.

Theme or Focal Area(s): Linear functions

In week 6, we discussed linear sequences and it was established that the general rule for a linear sequence is $u_n = a + (n - 1)d$ with the values of u_n and $(n - 1)$ changing from term to term. This presupposes that there are only two variables (u_n and $(n - 1)$) as the value of 'a' and d would be constant for all the terms of the same linear sequence. One of the features that qualifies $u_n = a + (n - 1)d$ to be a linear equation is the presence of exactly two variables

Linear functions are of the forms:

a. $f(x) = ax + b$: Variables are x and $f(x)$ while 'a' and 'b' are constants

b. $y = mx + c$: Variables are x and y while 'm' and 'c' are constants

c. $y - y_1 = m(x - x_1)$: Variables are x and y while y_1, m and 'x_1' are constants

Note: It must be noted that the degree of linear functions is one

Theme or Focal Area (s): Graphs of Linear Functions

Since linear functions have exactly two variables, any two-dimensional plane / system like the Cartesian coordinate system can be used to represent them. Corresponding values of the two variables are taken as pairs and plotted as locations on the grid. The infinitely many points that satisfy a linear function or equation join to form a straight line.

Some notes:

For a linear equation, $y = mx + c$,

1. The value of the y intercept is the y coordinate of the point where the graph of the function cuts the $y - axis$, when the value of $x = 0$
2. The value of the x intercept is the x coordinate of the point where the graph of the function cuts the $y - axis$, when the value of $y = 0$
3. m is the slope or gradient of the line. It is the ratio of the increment / decrement of values of the dependent variable to the increment / decrement of values of the independent variable. It often written as $m = \frac{\Delta y}{\Delta x}$ or $m = \frac{y_2 - y_1}{x_2 - x_1}$ or $m = \frac{y_1 - y_2}{x_1 - x_2}$ where (x_1, y_1) and (x_2, y_2) are on the line. Note that, $m \neq \frac{y_2 - y_1}{x_1 - x_2}$ since x_1 and y_2 are not corresponding coordinates just as x_2 and y_1 are not coordinates of the same point
4. If $m > 0$, the graph strictly increases and when $m < 0$, it strictly decreases

Example

To graph the equation $y = 2x - 1$, we can use

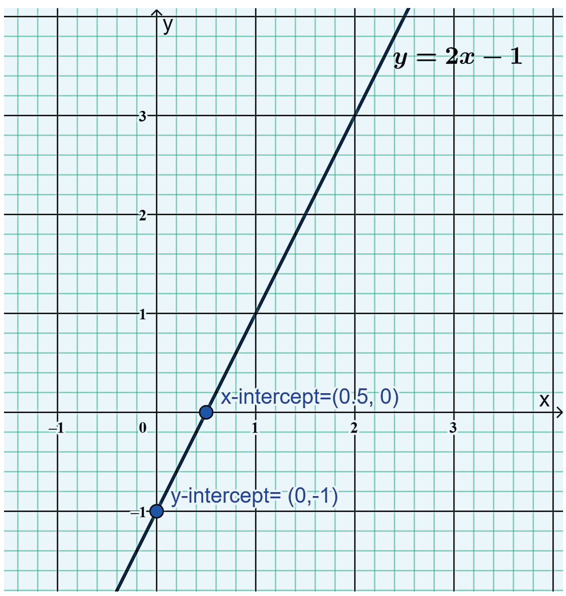
a. the intercepts method:

   i. Start by plotting the y-intercept: $(0, -1)$, which can be obtained by finding the value of y when $x = 0$

   ii. Plot the x-intercept: $(0.5, 0)$, which can be obtained by finding the value of x when $y = 0$

   iii. Connect the points to get a straight line;

***Note**: that since linear functions are polynomial functions and hence have the domain and ranges as the set of real numbers and the fact that a straight line has no extremities, it should be emphasized that lines drawn to connect the points should stretch to the ends of the grid*

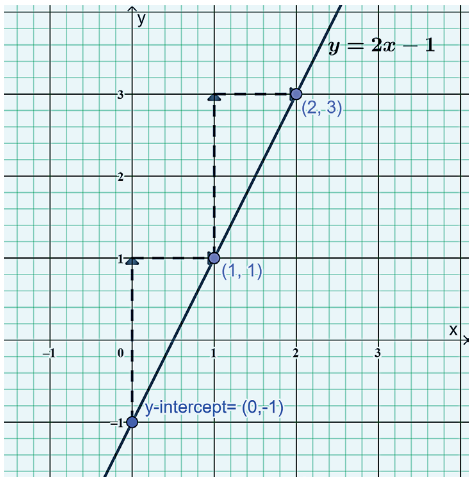


b. Using the gradient and *y*-intercept:

i. Start by plotting the *y*-intercept: $(0, -1)$, which can be obtained by finding the value of y when $x = 0$

ii. Use the slope, $m = 2$ to find other points. The slope can be found by comparing the equation to be graphed to the most appropriate general equation of a line, in this example, $y = mx + c$

iii. Since the slope is in the ratio 1 : 2, we can go up 2 units and right 1 unit from the *y*-intercept to find the next point $(0 + 1, -1 + 2) = (1, 1)$ and the next, $(1 + 1, 1 + 2) = (2, 3)$ and so on.

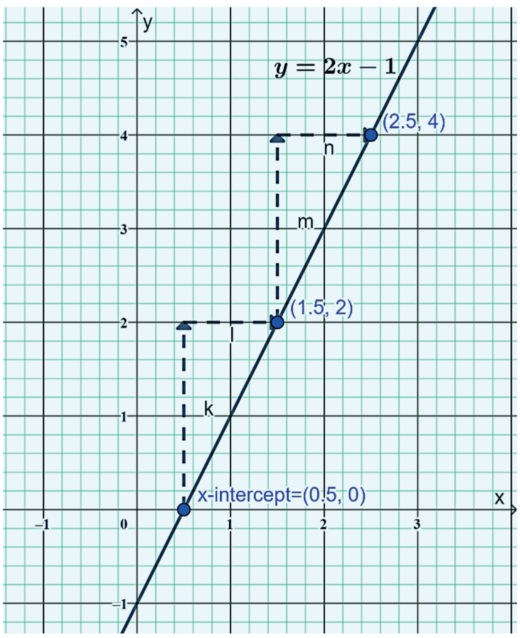


c. Using the gradient and *x*-intercept:

i. Start by plotting the *x*-intercept: $(0.5, 0)$, which can be obtained by finding the value of x when $y = 0$

ii. Use the slope, $m = 2$ to find other points.

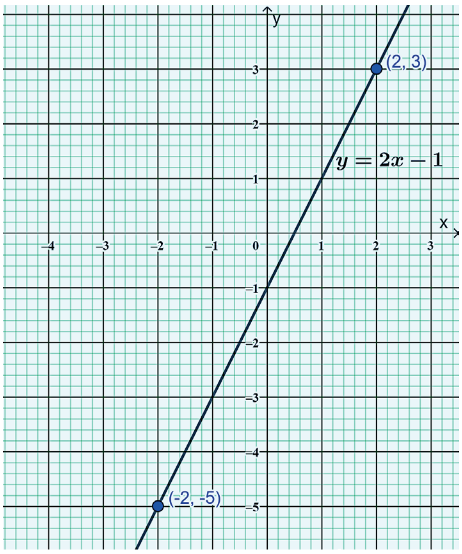
iii. Since the slope is in the ratio 1 : 2, we can go up 2 units and right 1 unit from the *y*-intercept to find the next point $(0.5 + 1, 0 + 2) = (1.5, 2)$ and the next and $(1.5 + 1, 2 + 2) = (2.5, 4)$ and so on.



d. Plotting points in a suitable range:

In some cases, it is not convenient to plot the intercepts of the line on the grid. Such situations can occur when multiple graphs have to be constructed on the same grid and a common scale for the axes that allows for all the intercepts to be located and easily plotted cannot be easily obtained. In a case such as this,

i. A convenient value for x may be chosen, say $x = 2$ and its corresponding y value ($y = 3$) found by substituting it into the equation of the line
ii. The process is repeated for a different value of x, say, $x = -2$ to obtain $(-2, -5)$
iii. The two points obtained can then be plotted and a straight line drawn to join the points to the edges of the grid



Theme or Focal Area(s): Determining Areas Enclosed By Graphs

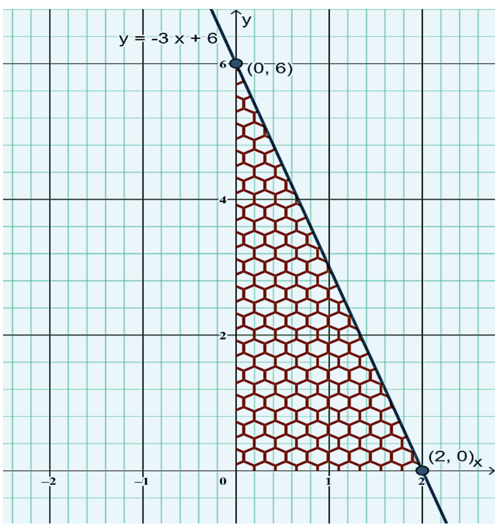
The area enclosed by two or more graphs can be found by finding the area of the plane figures that is bounded by the graphs.

Examples

a. Find the area bounded by $y = -3x + 6$, the $x - axis$ and the $y - axis$

Solution:

The graph of $y = -3x + 6$ intersects the $x - axis$ at (2, 0) and the $y - axis$ at (0, 6) while the two axes intersect at the origin with coordinates (0, 0)



The area bounded by $y = -3x + 6$, the $x - axis$ and the $y - axis$ is in the shape of a right triangle with vertices, (0, 0), (0, 6) and (2, 0).

This translates into a base length of 2 *units* and a height of 6 *units*.

The required area is thus: $\frac{1}{2}(2)(6) = 6$ *square units*

b. Consider the linear equation: $2x - 3y = 6$.

i. Draw the graph of this linear equation by hand or using a graphing utility on a coordinate plane.

ii. Find the area bounded by the graph, the $x - axis$, $y - axis$ and $y = 4$

Solution

For $2x - 3y = 6$,

$x - intercept = (3, 0)$

$y - intercept = (0, -2)$

Intersection points between $2x - 3y = 6$ and $y = 4$ is $(9, 4)$

From the graph in Figure 7, the bounded area (a trapezium) has vertices at $A(9, 4)$, $B(0, 4)$, $C(0, 0)$ and $D(3, 0)$

The parallel sides, $|AB|$ and $|DC|$ have lengths, 9 *units* and 3 *units* respectively while the perpendicular height is 4 *units*

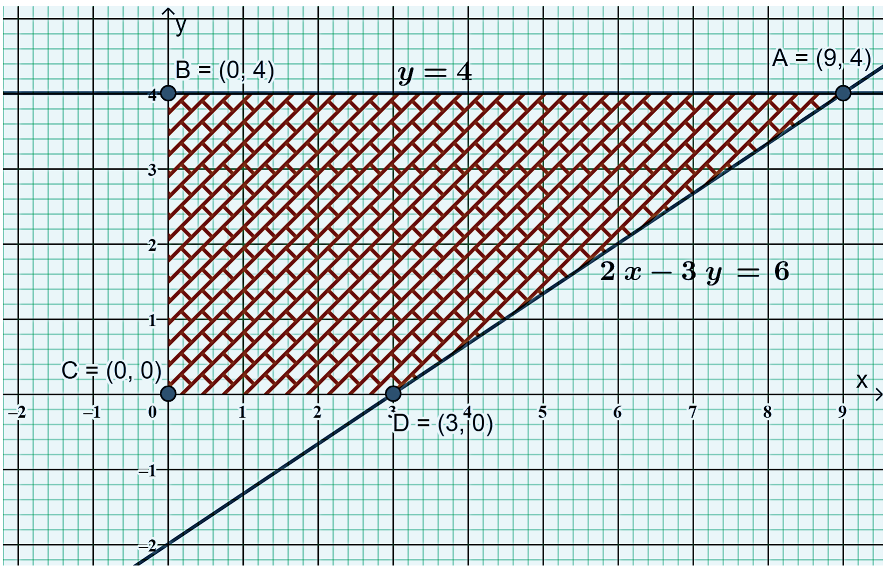


Figure 7

The area of the bounded region = *Area of ABCD* = $1/2(|AB| + |CD|) \times |BC|$

$= \frac{1}{2}(9 + 3)(4)$

$= 24$ *squared units*

Theme or Focal Area: Modelling with Linear Equations

Most real-life problems can be translated to the world of mathematics, solved and the solutions interpreted in the real life. Linear functions are among the wide range of mathematical tools for doing just that.

Examples

a. A local electronics store is selling a new smartphone model. The store manager has recorded the number of smartphones sold and the corresponding price for each week. The data is as follows:

Week 1: Price *GH¢* 800.00, Number of Smartphones Sold: 30

Week 2: Price *GH¢* 700.00, Number of Smartphones Sold: 40

Week 3: Price *GH¢* 600.00, Number of Smartphones Sold: 50

Week 4: Price *GH¢* 500.00, Number of Smartphones Sold: 60

Construct a linear function that represents the relationship between the price (p) and the number of smartphones sold (n) and predict the number of smartphones that would be sold if the price is reduced to *GH¢* 200.00

Solution

Price (p) / GH¢	800	700	600	500
Sales (n)	30	40	50	60

If the required function must be linear, then the pairs from the table must satisfy an equation of the form: $n = mp + c$

From the table,

When $p = 600$, $n = 50$. Thus

$50 = 600m + c$

Also, when $p = 500$, $n = 50$. Thus

$60 = 500m + c$

A system of two linear equations can be formed and solved to obtain the values of m and c as such:

$$\left.\begin{array}{l} 50 = 600m + c \\ 60 = 500m + c \end{array}\right\}$$

$-10 = 100m$

$m = -\frac{10}{100} = -0.1$ and

$60 = 500\left(\frac{10}{100}\right) + c$

$c = 60 + 50 = 110$

$\therefore$ the required function is $n(p) = -0.1p + 110$

$\therefore$ When $p = GH¢\ 200.00$, $n(200) = -0.1(200) + 110 = 90$

When the price is reduced to *GH¢* 200.00, 90 smartphones will be sold

Alternatively,

It can be observed from the table that there is a common difference in the prices i.e., *GH¢* 100.00 and a common difference in the number of sales too i.e., 10. This suggests that the ratio of difference in the codomain, range or values of N to the difference in the elements in the domain or values for price will be a constant. That constant is the gradient. The gradient is negative since we expect a decreasing function. While the prices increase, the number of sales decrease, thus:

$Gradient,\ m = -\frac{10}{10}0 = -0.1$

Linear functions are of the form, $y = mx + c$ so we expect the required function to be $n(p) = mp + c$

When $n = 60$, $p = 500$

$60 = -0.1(500) + c$

$c = 60 + 50$

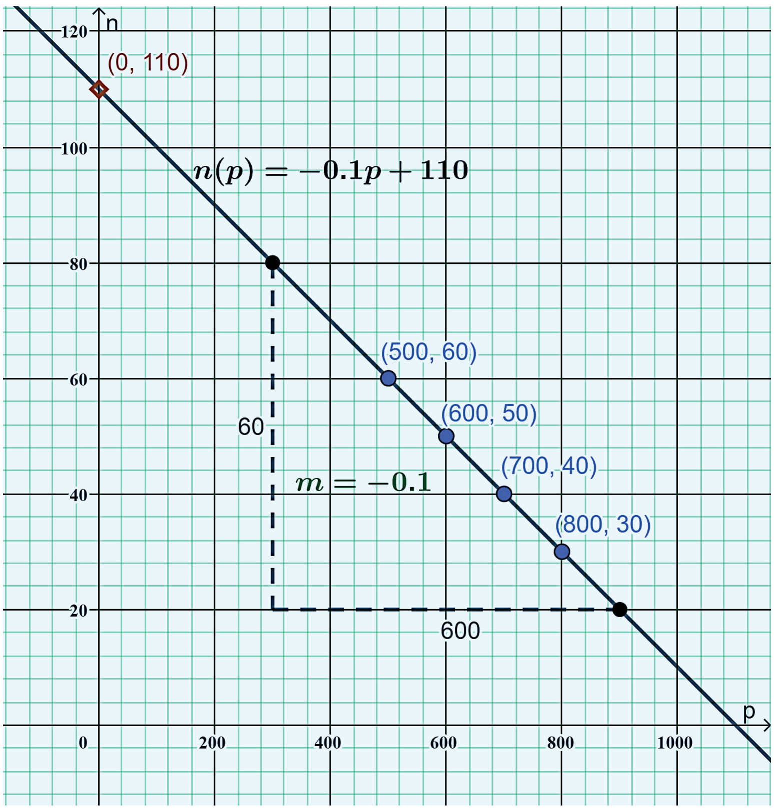
$c = 110$

$\therefore$ the required linear function is $n(p) = -0.1p + 110$

When $p =$ *GH¢* 200.00, $n(200) = -0.1(200) + 110 = 90$

$\therefore$When the price is reduced to *GH¢* 200.00, 90 smartphones will be sold

Another solution is to plot the points, (p, n) on the cartesian plane and use the graph to find the gradient and estimate the $y - intercept$ to obtain the values of m and c respectively as illustrated in



From the graph, when $p =$ *GH¢* 200.00, $n = 90$

$\therefore$When the price is reduced to *GH¢* 200.00, 90 smartphones will be sold

Theme or Focal Area(s): Parabolic or Quadratic Functions

As the name suggests, parabolic functions represent the path taken by points on a parabola.

Parabolic functions are of the forms $f(x) = a\,x\,2 + bx + c$ where a, b and c are constants and $a \neq 0$. The variables are x and $f(x)$ while 'a' and 'b' are constants. Parabolic functions have a degree of 2

Theme or Focal Area (s): Graphs of parabolic functions

The graph of a quadratic function in terms of x i.e., $y = a\,x\,2 + bx + c$ is either shaped like the intersection symbol, ∩ thus opening downwards or the union symbol, ∪ opening upward

Some notes:

For a parabolic function, $f(x) = a\,x\,2 + bx + c$ with vertex (turning point) located at (u, v)

1. If $a > 0$,

i. The graph of $f(x)$ is "⋃" shaped, same as the turning point
ii. It has a minimum point
iii. It decreases in the interval: $x = (-\infty, u)$ while increasing in the interval: $x = (u, \infty)$
iv. Its range is the interval: $y = [u, \infty)$
v. If $a < 0$
vi. The graph of $f(x)$ is "⋂" shaped
vii. It has a maximum point, same as the turning point
viii. It increases in the interval: $x = (-\infty, u)$ while decreasing in the interval: $x = (u, \infty)$
ix. Its range is the interval: $y = (-\infty, u]$

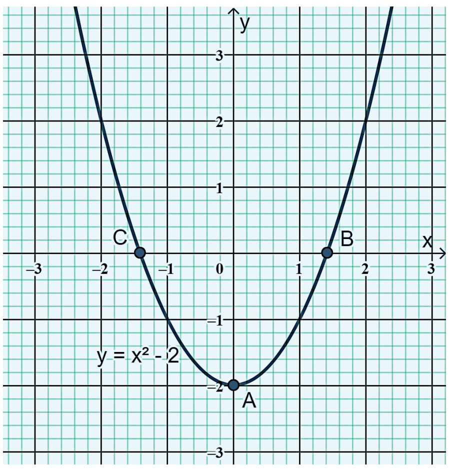


Illustration 1

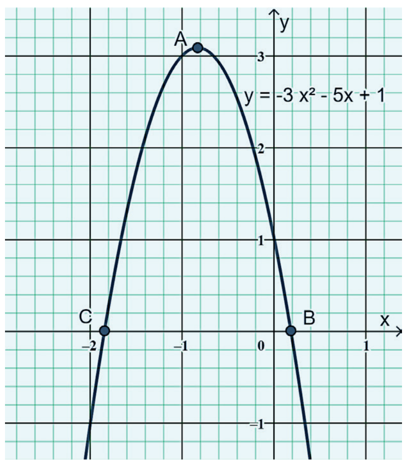


Illustration 2

Figure 8: *Parabolic graphs*

Illustration 1 in Figure 8 shows the graph of $y = x\,2 - 2$, having a positive number, 1 as the coefficient of the squared term. It has a minimum value at $A(0, -2)$, its turning point. It decreases for values of x in the interval: $(-\infty, 0)$ and increases for $0 < x < \infty$. $y \in [-2, \infty)$ as the values of y would not be less than -2, the y coordinate of the lowest point on the graph.

For $y = -3\,x^2 - 5x + 1$ (the corresponding graph shown in Illustration 2) however, the coefficient of the squared term, i.e., $-3 < 0$ and thus, is shaped like ⋂. It has a maximum value at $A(-0.833, 3.083)$, its turning point. It increases for values of x in the interval: $(-\infty, -0.833)$ and increases for $x \in (-0.833, \infty)$ and has a range between negative infinity to 3.0833 (inclusive) as the values of y cannot be greater than 3.0833.

Examples

1. Construct a parabolic function whose graph has a minimum value at its vertex at $(1, -2)$ and passes through the point $(2, 1)$ (0,1)

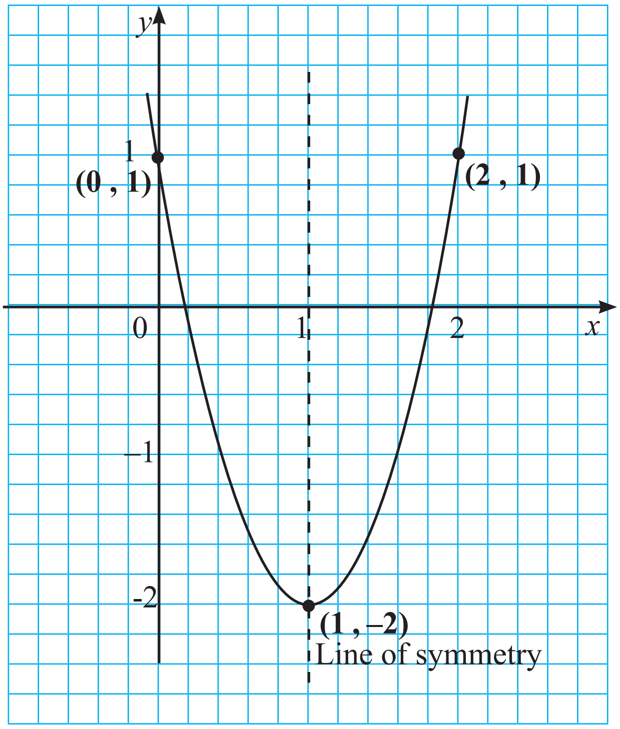
Solution:

Vertex is located at $(1, -2)$ (1,-2)

Point on graph $(2, 1)$

From the vertex, a movement of 3 *units* upwards and 1 *unit* to the right results in the location of $(2, 1)$

Since the graph of a parabolic function is bilaterally symmetrical, a movement from the vertex of 3 *units* upwards and 1 *unit* to the left will provide the location of another point $(1 - 1, -2 + 3) = (0, 1)$ which lies on the graph



We expect that any parabolic function to be of the form $f(x) = a\,x^2 + bx + c$ and since the coordinates of points that lie on its graph must satisfy the equation,

For $(1, -2)$,

$-2 = a\,(1)^2 + b(1) + c$

$a + b + c = -2$

For $(2, 1)$,

$1 = a\,(2)^2 + b(2) + c$

$4a + 2b + c = 1$

For $(0, 1)$,

$1 = a\,(0)^2 + b(0) + c$

$c = 1$

$$\left.\begin{aligned} a + b + c &= -2 \\ 4a + 2b + c &= 1 \\ c &= 1 \end{aligned}\right\}$$

The solution to the system of equations results in $a = 3$, $b = -6$ and $c = 1$ which are the constants in the quadratic function hence we obtain

$f(x) = 3x^2 - 6x + 1$

Learning Tasks

Guide learners to solve the following task and check for understanding. Support system should be provided to learners at lower ability levels.

Guide Learners to:

i. recognise and create linear and parabolic functions by hand and with technology where appropriate.

ii. use algebraic notions, properties of linear and non-linear functions, linear and non-linear equations and solve linear and non-linear simultaneous systems.

iii. solve linear equations in two variables find the solution using the graph and determine the area enclosed by the graphs.

iv. model statements into linear mathematical equations and solve simultaneously

Pedagogical Exemplars

This week's lessons seek to help Learners recognise and construct linear and parabolic functions by hand and with the aid of technology, and where appropriate show the intercepts and investigate the turning points. Learner will also be able to recognise and use appropriate algebraic notions, properties of linear and non-linear functions, linear and non-linear equations and solve linear and non-linear simultaneous systems. They will also be able to recognise linear equations in two variables and draw its graph by hand and by using the appropriate technology (e.g., GeoGebra, Demos, PhET Simulations, Geometer's Sketch Pad), find the solution using the graph and determine the area enclosed by the graphs. Finally, Learners will through this week's lessons be able recognise and model statements into linear mathematical equations and solve simultaneously as well.

The following pedagogical approaches are suggested:

1. **Review previous knowledge:** Review learners' previous knowledge on linear equations. Take note to make corrections and support learners who have little or no idea about the concepts.
2. **Using talk for learning:** engage and guide learners to construct linear and parabolic functions by hand and with the aid of technology. Make sure that enough examples of varying difficulty are provided to cater for the different group of learners to grasp the concept well.
3. **Using think-pair- share activity and mixed ability/gender groups:** engage learners through demonstrations and one-on-one discussions to recognise linear equations in two variables and draw its graph by hand and by using the appropriate technology (e.g., GeoGebra, Demos, PhET Simulations, Geometer's Sketch Pad), Give learners the opportunity to make group presentations (oral and written) with each member of the pair/group taking their turn to help in solving assigned task.
4. **In mixed ability groups:** Learners work in harmonised mixed-abilities groups to extend the idea of linear and parabolic functions to find solution using the graph and determine the area enclosed by the graphs to develop conceptual understanding. Give learners the platform to work out enough examples whiles you also address challenges faced by learners through one-on-one discussions.

5. **Using think- pair -share in mixed-ability groups**: In a well-controlled discussion, guide learners to model statements into linear mathematical equations and solve simultaneously. Encourage learners to share solutions to the whole class whiles you provide feedback which promotes further explanations on the strategies they used and how they arrived at their answers.
6. **In a well-regulated class discussion**: summarise the lesson for the week and give learners tasks to solve. Such tasks in the form of assignment or take-home tasks could be given to learners

Key Assessment

Assessment Level 1: Recall and Reproduction:

1. Construct a linear function passing through the points (1, 4) and (3, 10)
2. Given the linear function $y = 2x - 5$, find its x−intercept and y-intercept.

Assessment Level 2: Skills of Conceptual Understanding:

Draw graphs of the following functions, stating the y-intercepts, the minimum and maximum values, and the values for which the graph is increasing and decreasing:

a. $y = 2x - 3$,

b. $y = 3 - 2x$,

c. $y = 2x^2 + 3x - 1, -2 \leq x \leq 2$

d. $y = -2x^2 - 3x + 1, -2 \leq x \leq 2$

Assessment Level 3: Strategic reasoning:

Determine the turning point of the parabolic function $y = -2x^2 + 4x + 1$, state whether it is a maximum of minimum and describe the nature of its graph

Assessment Level 4: Extended critical thinking and reasoning:

1. Kojo and Afi went to the shop to buy stationery for school. Kojo bought 5 books and 3 pens for *GH¢*8. Afi bought 3 books and 4 pens for *GH¢*9. Find the cost of a book and a book.
2. A manufacturing company produces two types of products: Product *A* and Product *B*. The company's production costs and sales revenue for each product are as follows:

   Product *A*:

   Production Cost per unit: *GH¢*10

   Selling Price per unit: *GH¢*25

   Product B:

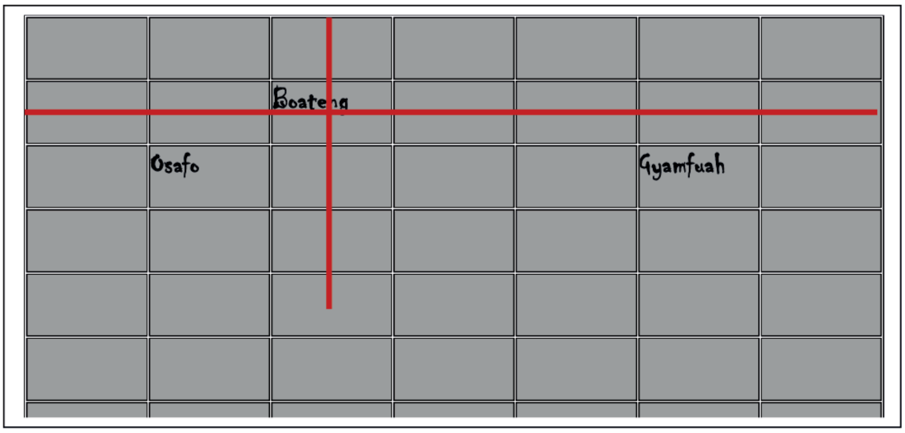
   Production Cost per unit: *GH¢*15

   Selling Price per unit: *GH¢*30

   In a given month, the company produced a total of 1,000 units of **Product *A*** and **Product *B*** combined. The total production cost was *GH¢*12,000, and the total sales revenue was *GH¢*27,000.

   a. Use appropriate algebraic notions and properties to represent the situation algebraically.
   b. Solve the simultaneous equations to find the number of units produced for each product
3. In an office, the allocation of a pigeon holes in a cabinet to Mr. Osafo, Ms. Gyamfuah and Mrs. Boateng is such that the spaces allocated to Mr. Osafo and Ms. Gyamfuah are located along the line that separates the upper section and the lower section while Mrs. Boateng's hole is located on the line that divides the cabinet into left and right sides as indicated in the picture below.

Find a quadratic function that describes the curve that passes through Mr. Osafo's and Ms. Gyamfuah's holes and has a maximum point.



Week 9

Learning Indicator(s): *Solve up to three systems of linear equations simultaneously by algebraic manipulations*

Introduction

This week's lessons introduce learners to fundamental concepts in systems of linear inequalities, and linear programming. These are crucial foundations in algebra and provide students with powerful problem-solving tools that are applicable across various disciplines. Learners would explore how to represent and solve systems of linear inequalities both graphically and algebraically. Understanding feasible regions and boundary lines becomes essential in interpreting solutions within practical contexts, such as budgeting, resource allocation, and geometric constraints.

Theme or Focal Area(s): Graphical solution to linear inequalities involving one variable

The solution to an inequality is the range of values for the variables involved that make the inequality true. For example, given $4x > 3$,

when $x = 1 \Rightarrow 4x = 4(1) = 4$, the inequality holds true since 4 is greater than 3

when $x = 3 \Rightarrow 4x = 4(3) = 12$, the inequality also holds true since 12 is greater than 3

there are infinitely many values for x which will make the inequality hold true hence it is more expedient to write out a range that caters for all those values. For $4x > 3$, all values of x which are greater than $\frac{3}{4}$ satisfy. This translates into a *truth set* $= \left\{x : x > \frac{3}{4}\right\}$ which can be illustrated by a shaded area on a graph as shown in Figure 9. It can be observed that a broken line is used to indicate the boundary of the solution region (shaded with blue). This is because the solution does not include $x = \frac{3}{4}$

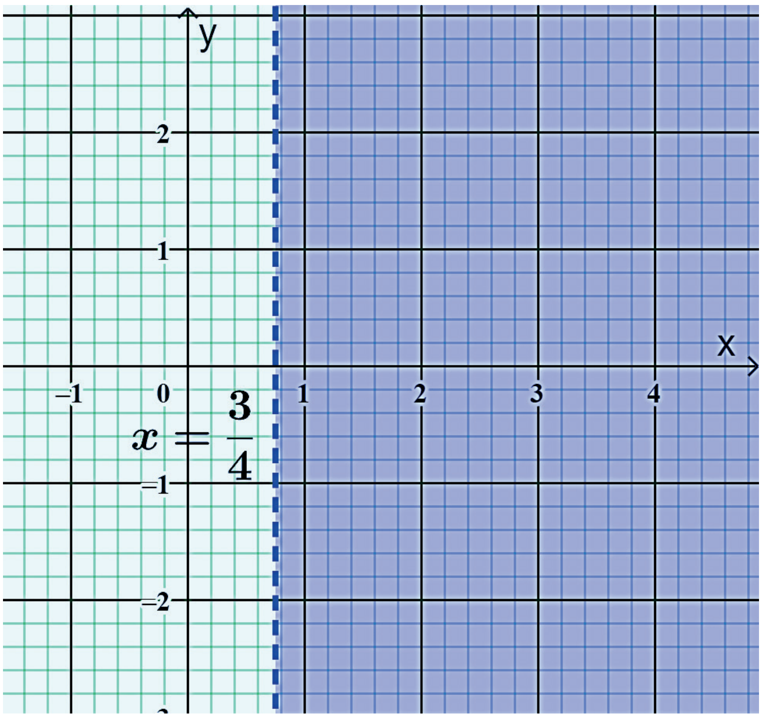


Figure 9: *Linear inequality in one variable*

Theme or Focal Area(s): Graphical solution to linear inequalities involving two variables

To show the graphical representation to a linear inequality involving two variables, say $2x - 3y < 4$,

1. the inequality is rewritten as an equation, i.e., $2x - 3y = 4$ so that the boundary of the solution can be drawn
2. a test to determine which side of the boundary line forms part of the solution region can then be conducted by determining which side of the line contains points that satisfy the inequality. Thus:
3. select a point, say (0, 0)
4. substitute the coordinates of the point into the inequality to check if it satisfies

   $2(0) - 3(0) = 0$

   0 is less than 4 hence (0, 0) satisfy the inequality and hence the side of the line, $2x - 3y = 4$ that forms the solution is the side that contains (0, 0) as shown in

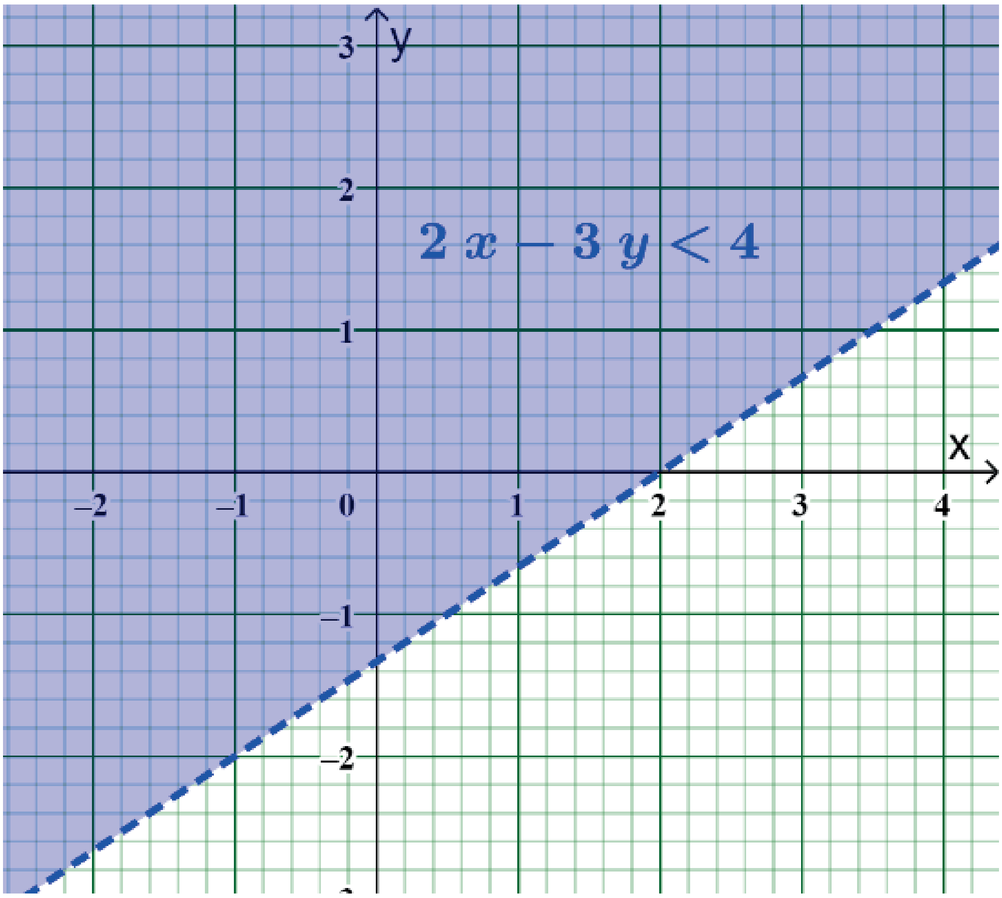


Figure 10: *Linear inequality in two variables*

Theme or Focal Area(s): Graphical solution to systems of linear inequalities

The solution region to a system of two or more linear inequalities is the region that is made up of all points that satisfy all the inequalities

Examples

1. Graph the solution set of the system

$x + 2y \geq 8$

$x + y \leq 12$

$x \geq 0$

$y \geq 0$

Solution:

We will go through the following steps to find the solution region

a. draw the graphs of $x + 2y = 8$, $x + y = 12$, $x = 0$ and $y = 0$,

b. identify regions on the grid that contain points that may satisfy all inequalities,

c. verify whether or not test points from those regions satisfy the inequalities

d. make conclusions to determine the solution region

Figure 11 shows the graphs of the four equations and possible solution regions namely A_1, A_2 and A_3. We would only consider points in the regions named A_1, A_2 and A_3 since points in those regions satisfy both $x \geq 0$ and $y \geq 0$

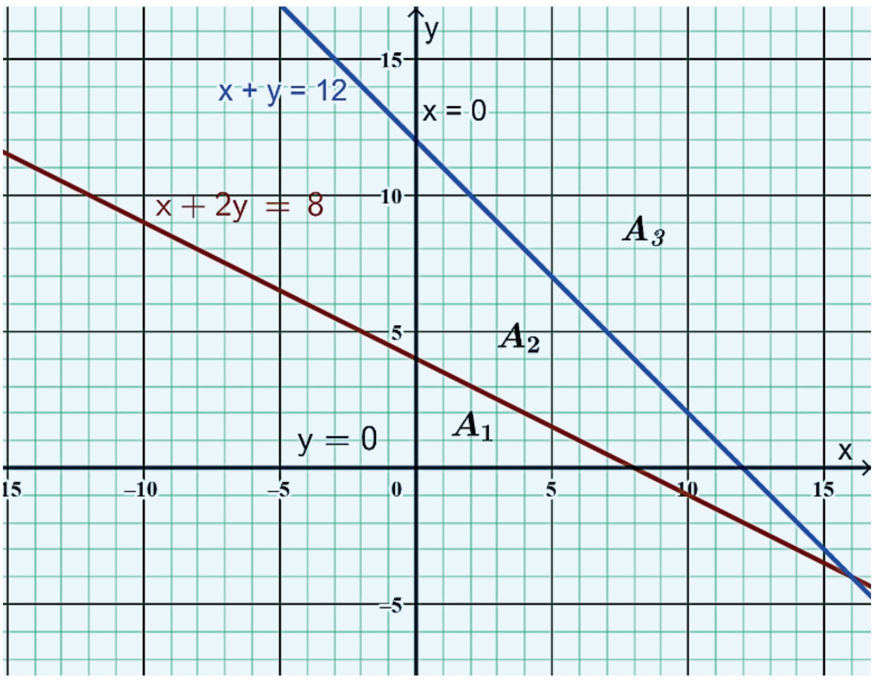


Figure 11

Region	Test Point	Inequality	Check	Conclusion
A_1	(1, 1)	$x + 2y \geq 8$	$1 + 2(1) = 3$	Does not satisfy
		$x + y \leq 12$	$1 + 1 = 2$	Satisfies
A_2	(5, 5)	$x + 2y \geq 8$	$5 + 2(5) = 15$	Satisfies
		$x + y \leq 12$	$5 + 5 = 10$	Satisfies
A_3	(10, 10)	$x + 2y \geq 8$	$10 + 2(10) = 30$	Satisfies
		$x + y \leq 12$	$10 + 10 = 20$	Does not satisfy

From the table, A_2 is the solution region since it is the only region which contains points that satisfy all four inequalities

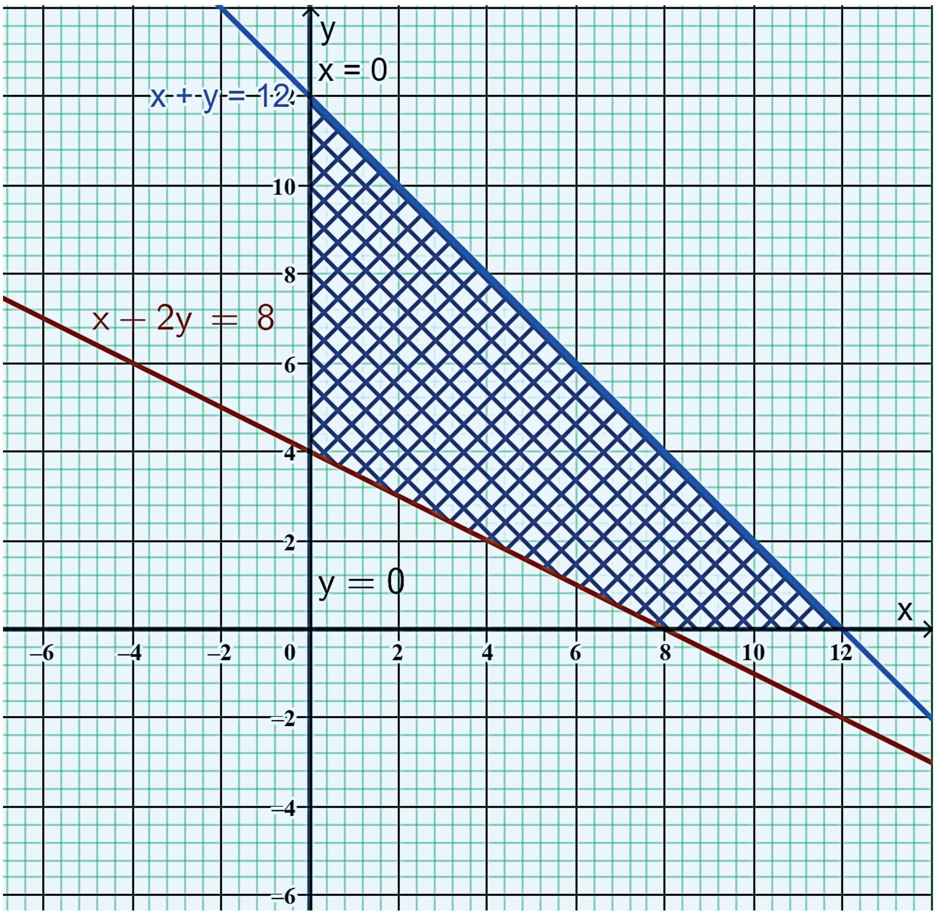


Figure 12

Theme or Focal Area(s): Linear programming

Linear programming introduces learners to optimization techniques within a system of linear constraints. They will learn to maximize or minimize an objective function while adhering to given constraints. Through graphical and algebraic methods, students gain insights into finding optimal solutions to real-world problems, such as production planning, transportation logistics, and financial portfolio management. Hands-on activities, interactive demonstrations, and real-world case studies foster critical thinking, problem-solving skills, and a deeper appreciation for the relevance of mathematics in various fields of study and future careers. As students grasp these concepts, they not only enhance their mathematical proficiency, but they also develop essential analytical skills necessary for success beyond the classroom.

Example

1. A factory produces two agricultural pesticides, A and B. For every barrel of A, the factory emits 0.25 kg of carbon monoxide (CO) and 0.60 kg of sulfur dioxide (SO_2), and for every barrel of B, it emits 0.50 kg of CO and 0.20 kg of SO_2. Pollution laws restrict the factory's output of CO to a maximum of 75 kg and SO2 to a maximum of 90 kg per day.
2. Find a system of inequalities that describes the number of barrels of each pesticide the factory can produce and still satisfy the pollution laws. Graph the feasible region.
3. Would it be legal for the factory to produce 100 barrels of A and 80 barrels of B per day?
4. Would it be legal for the factory to produce 60 barrels of A and 160 barrels of B per day?

Solution:

a. We will first record the data into a table

	CO / kg	$S\,O_2$ / kg
A	0.25 kg	0.60 kg
B	0.50 kg	0.20 kg
	75 kg	90 kg

Let x = *number of barrels of A produced per day*

y = *number of barrels of B produced per day*

The number of barrels produced cannot be negative hence $x \geq 0$ and $y \geq 0$ and since there are maximum masses of CO and $S\,O_2$ that are allowed,

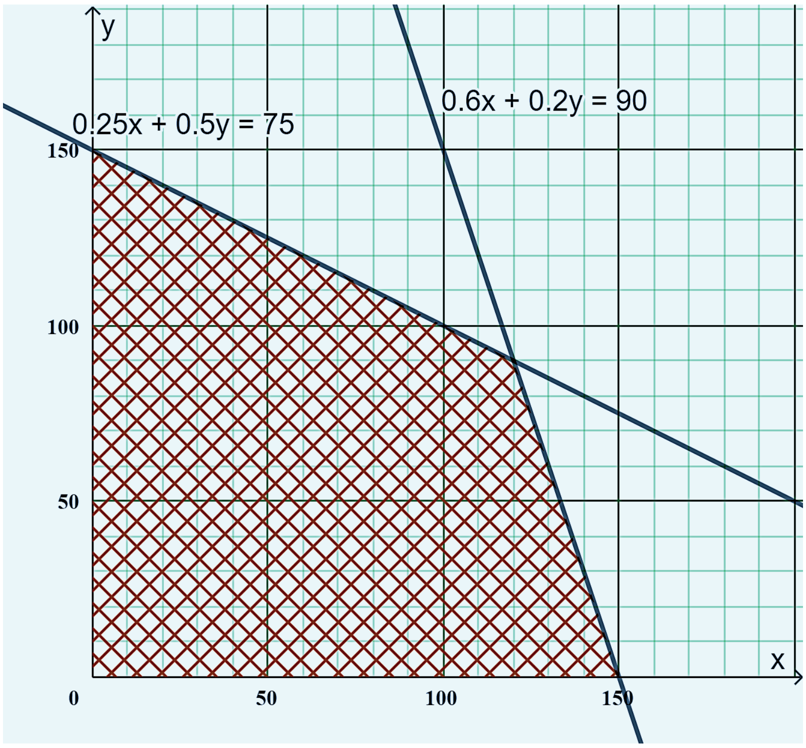
$\textit{Total amount of CO produced} = 0.25x + 0.50y \leq 75$

$\textit{Total amount of } S\,O_2 = 0.60x + 0.20y \leq 90$

These bits of data translate into the system:

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 0.25x + 0.50y &\leq 75 \\ 0.60x + 0.20y &\leq 90 \end{aligned}\right\}$$

Test Point	Inequality	Check	Conclusion
(50, 50)	$0.25x + 0.50y \leq 75$	$0.25(50) + 0.5(50) = 37.5$	Satisfies
	$0.60x + 0.20y \leq 90$	$0.60(50) + 0.20(50) = 40$	Satisfies
(50, 200)	$0.25x + 0.50y \leq 75$	$0.25(50) + 0.50(200) = 112.5$	Does not satisfy
	$0.60x + 0.20y \leq 90$	$0.60(50) + 0.20(200) = 70$	Satisfies
(150, 50)	$0.25x + 0.50y \leq 75$	$0.25(150) + 0.50(50) = 62.5$	Satisfies
	$0.60x + 0.20y \leq 90$	$0.60(150) + 0.20(50) = 100$	Does not satisfy



Note that the equations could be multiplied through to a convenient number to obtain an equivalent equation with whole numbers as coefficients. E.g.:

$0.25x + 0.50y \leq 75 \equiv x + 2y \leq 300$

b. (100, 80) falls in the solution region hence it is legal for the factory to produce 100 barrels of *A* and 80 barrels of *B* per day

c. It would be illegal for the factory to produce 60 barrels of A and 160 barrels of B per day since (60, 160)

Learning Tasks

Guide learners to solve the following task and check for understanding. Support system should be provided to learners who will struggle.

Learners to

a. review the concept of linear inequalities and systems of linear inequalities.

b. find graphically, solutions to linear inequalities involving one and two variables.

c. find graphically solution to systems of linear inequalities.

d. f ind optimal solutions to real-world problems using linear programming.

Pedagogical Exemplars

The aim of the lessons for this week is for all learners to be able to solve up to three systems of linear equations simultaneously by algebraic manipulations. The following are suggested activities for learning facilitation:

Review previous knowledge: Review learners' previous knowledge on linear inequalities involving one and extend it to two variables. Take note to make corrections and support learners who have little or no idea about the concepts.

1. **In convenient groups (ability, mixed ability, mixed gender, or pairs etc.):** Guide learners**:**
   a. to identify graphs which are linear and those which are non-linear.
   b. determine areas enclosed by graphs of linear and non-linear functions using the idea of liner programming
   c. solve up to three systems of linear equations involving three unknowns.
   d. solve systems of linear equations using elimination and substitution of other appropriate algebraic methods
   e. solve real-life problems involving simultaneous linear equations
2. **Using talk for learning:** engage and guide learners to show the graphical representation to linear inequalities involving one variable. Make sure that enough examples of varying difficulty are provided to cater for the different group of learners to grasp the concept well.
3. **Using think-pair- share activity and mixed ability/gender groups:** engage learners through demonstrations and one-on-one discussions to draw the boundary of solution to linear inequalities and determine which side of the boundary line forms part of the solution region. Give learners the opportunity to make group presentations (oral and written) with each member of the pair/group taking their turn to help in solving assigned task.
4. **In mixed ability groups:** Learners work in harmonised mixed-abilities groups to extend the idea of graphical solution to linear inequalities involving one and two variables to systems of linear inequalities. Give learners the platform to work out enough examples whiles you also address challenges faced by learners through one-on-one discussions.
5. **Using think- pair -share in mixed-ability groups**: In a well-controlled discussion, guide learners to solve real-world problems involving optimization techniques in production planning, transportation logistics, and financial portfolio management using linear programming. Encourage learners to share solutions to the whole class whiles you provide feedback which promotes further explanations on the strategies they used and how they arrived at their answers.

Key Assessment

Level 2 Skills of conceptual understanding:

1. Graph the solution of the following systems of inequalities, find the coordinates of all vertices and determine whether the solution set is bounded or not

a. $\begin{cases} x > 2 \\ y < 12 \\ 2x - 4y > 8 \end{cases}$

b. $\begin{cases} y < x + 6 \\ 3x + 2y \geq 12 \\ x - 2y \leq 2 \end{cases}$

Level 3 Strategic reasoning:

1. Suppose you are asked to solve a system of two linear equations in two variables.
2. Would you prefer to use the substitution method or the elimination method?
3. How many solutions are possible? Draw diagrams to illustrate the possibilities?

Level 4 Extended critical thinking and reasoning:

1. A publishing company publishes a total of no more than 100 books every year. At least 20 of these are nonfiction, but the company always publishes at least as much fiction as nonfiction. Find a system of inequalities that describes the possible numbers of fiction and nonfiction books that the company can produce each year consistent with these policies. Graph the solution set
2. Kwame only needs corn dough and water to prepare "koko" for his younger siblings. While some of his siblings prefer their "koko" light, others prefer theirs thick. Kwame realises that it requires 600g of corn dough and 1 cup of water to prepare light "koko" while 750 g of corn dough and 1 1⁄2 cups of water can be used to prepare thick "koko" the way his siblings want it. If there is just 2 kg of corn dough and 6 cups of water available, find a system of inequalities that describes the possible number of cups of light and thick "koko" that can be prepared. Graph the solution set.

Week 10

Learning Indicator(s):

1. *Describe polynomial functions and perform the basic arithmetic operations on them*
2. *Use the method of completing the square to transform any quadratic equation in, say x into an equation of the form (x − p) 2 = q that has the same solutions and explain how the quadratic formula is derived from this form*
3. *Use the remainder and factor theorems to find the factors and remainders of a polynomial of degree not greater than 4*
4. *Draw the graph of a polynomial function with degree up to 3 by hand and by using technology (e.g., GeoGebra, Demos, PhET Simulations, and Geometer's Sketch Pad) where appropriate*

Definition/Introduction

1. Polynomial functions comprise various combinations on constants, variables, and exponents and is of the form: $P_n(x) = a\,x^n + b\,x^{n-1} + c\,x^{n-2} + d\,x^{n-3} + \ldots$ where a, b, c, d, … are constants and n is a non-negative integer
2. Some types of polynomial functions are, linear, quadratic, cubic, quartic and quintic functions. The classification of polynomial functions into these types depends on the degree (the highest exponent of the independent variable) of the polynomial. For example, the highest exponent of x in $-2\,x^2 + 4\,x^3 + 2x$ is 3 and thus, the expression is a cubic polynomial expression

n	Type of polynomial	General Form	Examples
0	Constant	$f(x) = c$	$y = 2,\ y = -4,\ y = \frac{2}{3}$
1	Linear	$f(x) = ax + b,\ ,\ a \neq 0$	$f(x) = -2x + 3,\ y = 3x - 4,\ x - 2y = 4$
2	Quadratic	$f(x) = a\,x^2 + bx + c\ ,\ a \neq 0$	$f(x) = 2\,x^2 - 4x + 3,\ y = -5\,x^2 - 4,$ $g(x) = x^2 + \frac{3}{2}\,x$
3	Cubic	$f(x) = a\,x^3 + b\,x^2 + cx + d\ ,$ $a \neq 0$	$y = x^3,\ y = x^3 - 2x,\ f(x) = 3 + 4x - x^3$
4	Quartic	$f(x) = a\,x^4 + b\,x^3 + c\,x^2 + dx +$ $g,\ a \neq 0$	$y = -x^4 + x^3,\ y = 3\,x^4 + x^3 - 2x,$ $f(x) = 3 + 4\,x^2 - x^3 - x^4$
5	Quintic	$f(x) = a\,x^5 + b\,x^4 + c\,x^3 +$ $d\,x^2 + gx + h,\ a \neq 0$	$f(x) = -2\,x^5 + 3,\ y = x^5 + 3x - 4,$ $x^5 + x^3 - 2y = 4$

Theme or Focal Area (s): Performing arithmetic operations on polynomial functions

Only like terms (terms which have the same variables raised to the same powers) can be added or subtracted.

Example 1:
If $f(x) = 4x^4 - x^3 - 9x^2 + 2x - 5$ and $h(x) = x^3 + 3x^2 - 2x + 2$ then,

a. $f(x) + h(x) = 4x^4 - x^3 - 9x^2 + 2x - 5 + (x^3 + 3x^2 - 2x + 2)$

$= 4x^4 - 6x^2 - 3$

b. $f(x) - h(x) = 4x^4 - x^3 - 9x^2 + 2x - 5 - (x^3 + 3x^2 - 2x + 2)$

$= 4x^4 - 2x^3 - 12x^2 + 4x - 7$

Example 2:
Divide $x^3 - 5x^2 + 4x - 3$ by $x - 2$

Solution

Long Division Method

$$\begin{array}{r} x^2 - 3x - 2 \\ x - 2 \;\sqrt{x^3 - 5x^2 + 4x - 3} \\ -\underline{(x^3 - 2x^2 + 0x + 0)} \\ -3x^2 + 4x - 3 \\ -\underline{(-3x^2 + 6x - 0)} \\ -2x - 3 \\ -\underline{(-2x + 4)} \\ -7 \end{array}$$

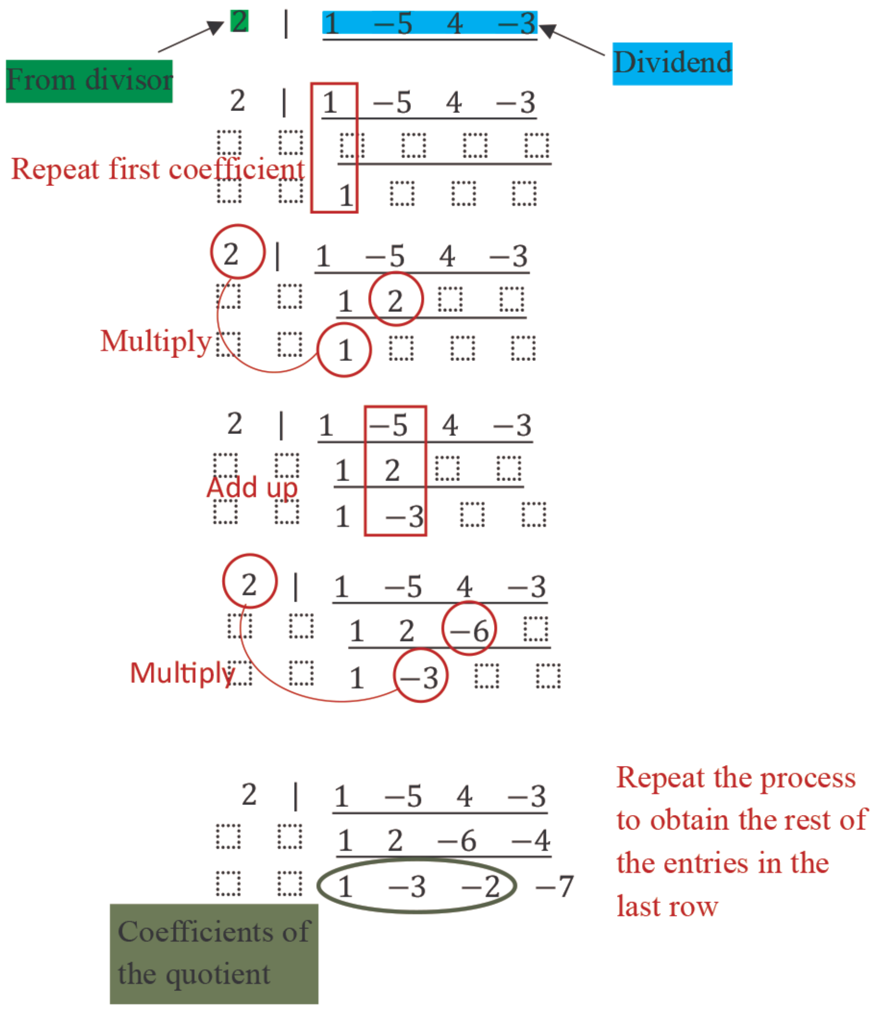
$\frac{x^3 - 5x^2 + 4x - 3}{x - 2} = x^2 - 3x - 2 - \frac{7}{x - 2}$ where $x^2 - 3x - 2$ is the quotient and -7 is the remainder

Synthetic Method

We begin by writing the coefficients to represent the divisor and the dividend

$Divisor = x - 2$

$x - 2 = 0 \Rightarrow x = 2$



Since the coefficients of the quotients are only three, it suggests that there are only three terms which are x 2, $-3x$ and -2. -7 is the remainder of the division

$$\therefore \frac{x^3 - 5x^2 + 4x - 3}{x - 2} = x^2 - 3x - 2 - \frac{7}{x - 2}$$

Theme or Focal Area(s): Polynomial graphs

Constant Functions

Constant functions of the form $y = c$ or $x = c$ represent horizontal or vertical lines.

For $y = c$, the value of y (which indicates the height of the graph) remains constant regardless of the value of x. c indicates the value of y where the line cuts the $y - axis$

The graph of $x = c$ is a vertical line that cuts the $x - axis$ when $x = c$. Figure 12 shows the graphs of $x = -3$, $x = 2$, $y = -2$ and $y = 5$

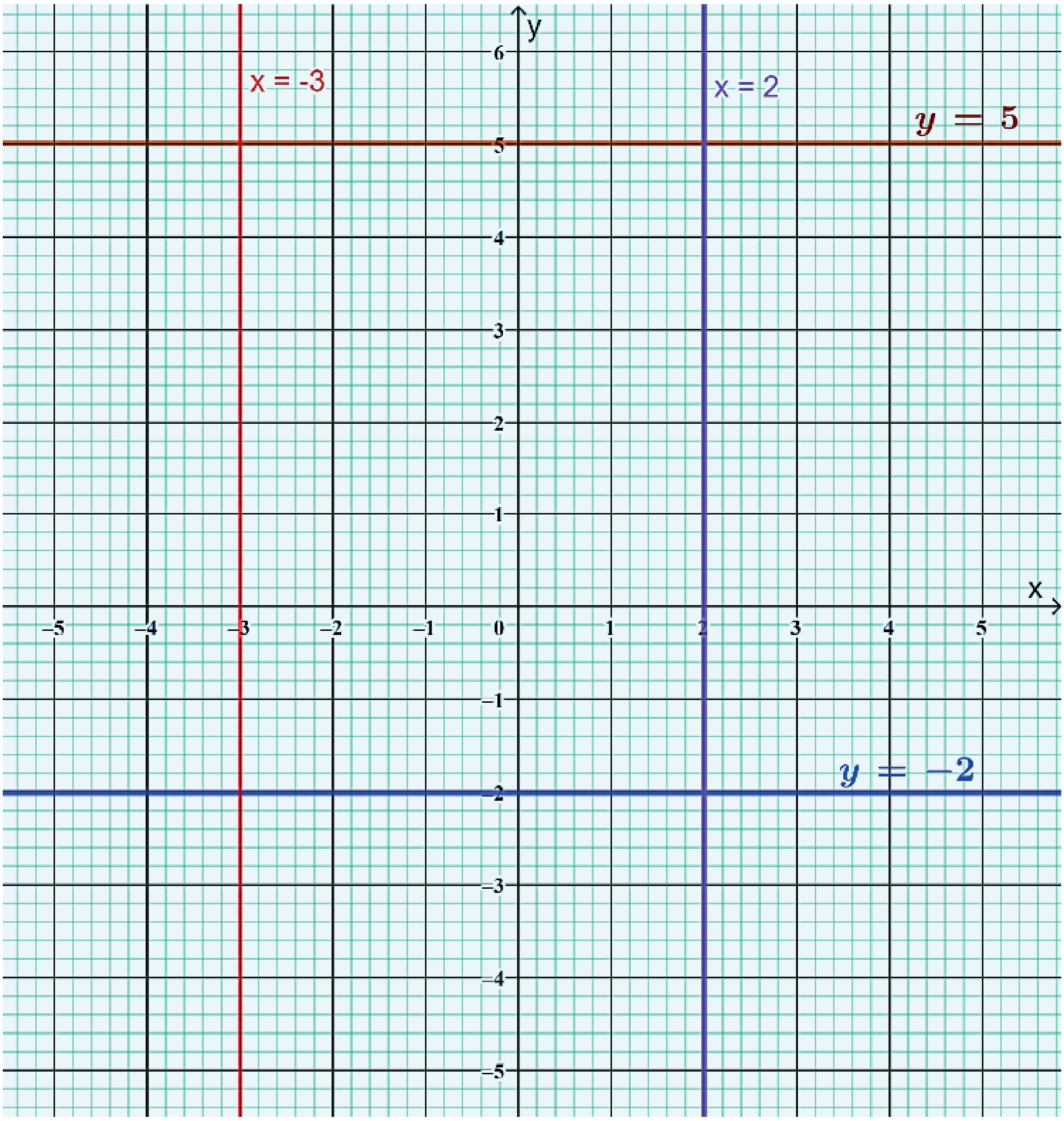


Figure 12: *Graphs of constant functions*

The lines $x = 0$ and $y = 0$ are special lines i.e., they coincide with the $y-axis$ and the $x-axis$ respectively

Linear Functions

$f(x) = ax + b$ represents a straight line which cuts the $y - axis$ at 'b' and has slope, 'a'. The value of 'a ', i.e., negative of positive, determines the nature of the straight line. If $a < 0$, the straight line strictly decreases and if $a > 0$, the line strictly increases.

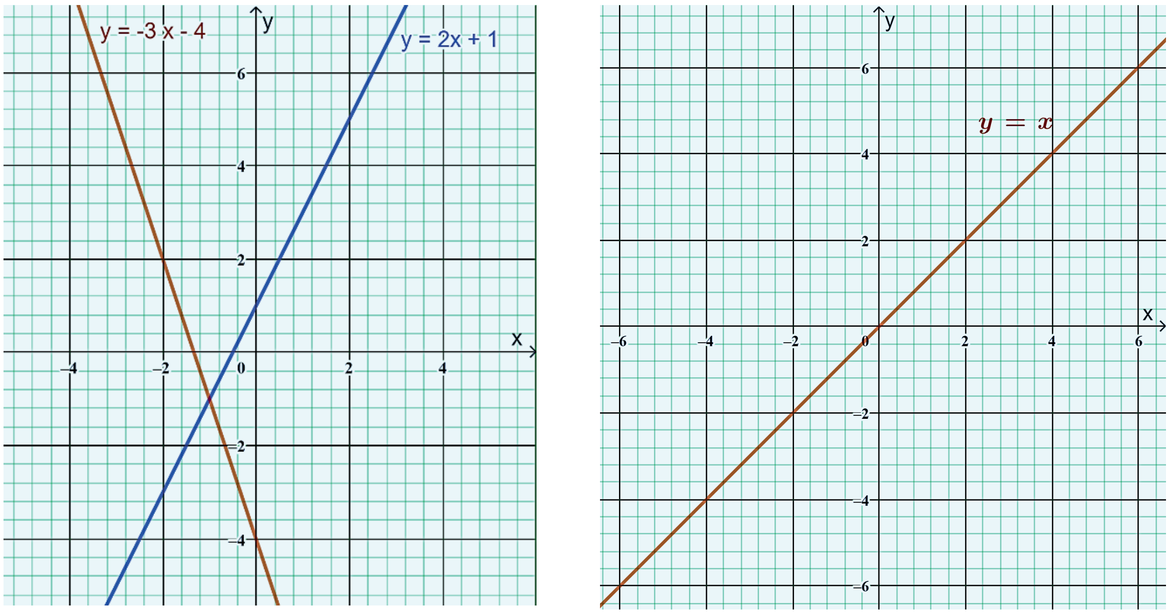


Illustration 1 ***Illustration 2***

Figure 13: *Graphs of linear functions*

The basic linear function with equation $y = x$ is the straight line that bisects the Cartesian plane as shown in illustration 2 in Figure 13. It passes through all points that have the same x and y coordinates including the origin.

Quadratic Functions

Quadratic functions are also called parabolic functions because their graphs are parabolas. Graphs of parabolic functions have been discussed extensively in week 8

Cubic Functions

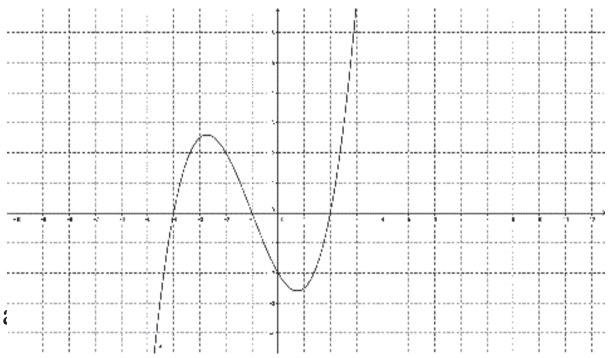
Parabolic graphs can be used as a model to graph cubic functions.

Example

$$f(x) = \frac{x^3}{4} + \frac{3x^2}{4} - \frac{3x}{4}2 - 2$$

$$= \frac{1}{4}(x^3 + 3x^2 - 6x - 8)$$

$$= \frac{1}{4}(x+4)(x+1)(x-2)$$



The roots of the graph are $x = -4$, $x = -1$ and $x = 2$

The graph increases in the interval $(-\infty, -2.7)$ (-2.7, 0.73)

Theme or Focal Area(s): Completing the squares

The quadratic formula and its usage.

1. Use the method of completing the squares to write the general quadratic functions $a x^2 + bx + c = 0$ in the form

   $A\ (x \pm B)^2 \pm C = 0$ or $A\ (x \pm B)^2 = C$, where A, B and C are constants, and C is the maximum or minimum value of the function, and the maximum and minimum occurs at $x \pm B = 0$.

2. Recognise that the expression $b^2 - 4ac$ of the general quadratic formula:

   $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ determines the nature of the roots of the function and the following properties holds:

   a. If $b^2 - 4ac \geq 0$, the roots are real
   b. If $b^2 - 4ac > 0$, the roots are real and different/discrete
   c. If $b^2 - 4ac < 0$, the roots are complex and imaginary
   d. If $b^2 - 4ac = 0$, the roots are real and equal or the equation is a perfect square.

Use of quadratic equation in real-life situation

Quadratic equations are important and widely used in various fields, including mathematics, physics, engineering, economics, and computer science. Some common applications and uses of quadratic equations include:

a. Solving for unknowns: The primary use of quadratic equations is to find the values of the variable x that satisfy the equation. This is especially useful when dealing with problems that involve motion, distance, area, or quantities that vary quadratically.

b. Geometry: Quadratic equations are often used in geometry to find the coordinates of points, lengths of sides, or angles of shapes.

c. Projectile motion: When an object is thrown or launched into the air, its path can often be modelled by a quadratic equation, helping to determine its maximum height, time of flight, and range

Use the roots (sum and products) to find other quadratic equations.

Explore relationship between the constants a, b *and* c of the general quadratic function $ax^2 + bx + c = 0$ and α *and* β, and use these relations to write other quadratic equations with given roots.

Example

Find the quadratic equation whose roots are −2 and 3 and has a minimum value

Solution:

$x^2 - (sum\ of\ roots)x + (products\ of\ roots) = 0$

$x^2 - (-2 + 3)x + (-2)(3) = 0$

$x^2 - x - 6 = 0$

Theme or Focal Area: Factor and remainder theorems

Use the factor and remainder theorems to solve problems of polynomial functions in context:

a. explore and recognize that generally for any polynomial function $P(x)$ division by another polynomial result in:

$$\frac{Dividend}{Divisor} = quotient + R\frac{emainder}{Divisor}$$

and so;

$$Dividend = Quotient \times Divisor + Remainder$$

b. Learners recognize that for any polynomial functions $P(x)$, if $(x - a)$ is a divisor then;

$P(x) = q(x) \cdot (x - a) + r$; where $q(x)$ is the quotient and r is the remainder.

c. Establish the Remainder theorem and use it to solve related problems:

Remainder Theorem: Given any polynomial functions $P(x)$, if $(x - a)$ is a divisor then;

$P(a) = q(a) \cdot (a - a) + r$; and $P(a) = r$

Example

Find the remainder when the polynomials $f(x) = 4x^4 - x^3 - 9x^2 + 2x - 5$ is divided by $(x - 2)$

Solution:

$x - 2 = 0$

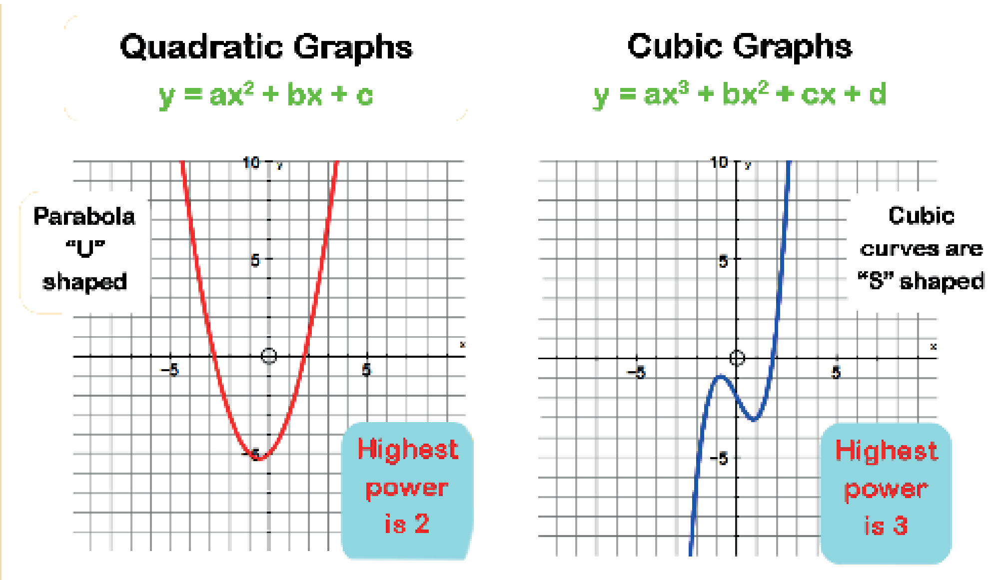
$\Rightarrow x = 2$

$f(2) = 4(2)^4 - 2^3 - 9(2)^2 + 2(2) - 5$

Theme or Focal Area: Graphs of polynomial functions

Use graphs of polynomial functions to solve problems.

a. Draw quadratic and cubic polynomial functions by hand and with the aid of appropriate technology and discuss what happens to the curve as $x - values$ decreases or increases, the point where the curve touches the $y - axis$, axis of symmetry and turning points.



Learning Task

Guide learners to solve the following task to check for understanding. Support system should be offered to learners who will struggle.

Learners to

a. review the concepts about linear and non-linear functions.

b. solve arithmetic operations on polynomial functions.

c. recall facts about quadratic and polynomial functions.

d. draw graphs of polynomial function by hand and by using ICT tools.

Pedagogical Exemplars

The objective of the lessons for this week is for all learners to recognise and describe polynomial functions and perform the basic arithmetic operations on them. Learners are expected to use the method of completing the square to transform quadratic equations. They are to apply the remainder and factor theorems to find the factors and remainders of a polynomial of various degrees. Learners are also to be able to draw the graph of a polynomial function. The following pedagogical exemplars strategies are suggested in the curriculum to guide lesson administrations:

a. **Review previous knowledge:** Recall key facts about linear and non-linear functions using talk-for-learning approach.

b. **In collaborative and well-supervised groups:** Guide learners to explore arithmetic operations (addition, subtraction, multiplication and division) on polynomial functions.

c. **Use problem-based approach:** to guide Learners to come out with two or more polynomial functions with degree up to 5.

d. **Using talk-for-learning:** guide learning to recall key facts about the quadratic equation $ax^2+ bx + c = 0$

e. **In collaborative and well-supervised groups:** guide Learners to explore relationship between the constants a, b and c of the general quadratic equation $ax^2 + bx + c = 0$ and α and β, and use these relations to write other quadratic equations with given roots.

f. **Using talk-for-learning:** help Learners recall key facts about polynomial functions.

g. **Experiential learning:** By use of hand and leveraging on technology, guide learners to draw graphs of polynomials up to degree 3.

Key Assessment

Assessment level 1: Recall:

1. Given $g(x) = 3x^2 - 5x + 2$ and $h(x) = 2x^2 + 4x - 1$

a. Determine the sum of $g(x)$and $h(x)$

b. Find the difference between $g(x)$and $h(x)$

2. Consider the polynomial function $p(x) = 6x^4 - 3x^3 + 2x^2 - 8x + 5$

a. What is the degree of the polynomial function?

b. What is the leading coefficient of the polynomial?

c. Evaluate $p(1)$

Level 2 Skills of conceptual understanding:

a. Given the polynomial $f(x) = x^3 - 2x^2 + 3x - 1$, use the Remainder Theorem to find the remainder when $f(x)$ is divided by $x - 2$

b. Given the polynomial $p(x) = 3x^4 - 4x^3 + 2x^2 - 7x - 10$, use the Remainder Theorem to determine if $p(x)$ is divisible *by* $x - 2$

Level 3 Strategic reasoning:

1. A local farmer wants to build a rectangular pen for his animals. He plans to use one side of his barn as one side of the pen, and he has 200 *feet* of fencing material to complete the other three sides. What dimensions should the farmer use to maximize the area of the pen?

Week 11

Learning Indicator(s):

1. *Recognise a rational function and determine the domain and range*
2. *Carry out the basic arithmetic operations on rational functions*
3. *Apply partial fraction decomposition up to factors with exponents and irreducible quadratic factors*

Theme or Focal Area(s): Definition, domain, range and zeros of rational functions

A rational function, much like a rational number (a fraction of integers), consists of a **ratio between two polynomial expressions**. The denominator cannot be zero.

A rational function is of the form

$R(x) = f(x)$ _ $g(x)$; $g(x) \neq 0$ (That is, a ratio of two polynomials for which the divisor is not zero).

For example, $f\left(x\right) = \frac{x^2+1}{x-2}$ is a rational function.

The concept of rational functions finds their use in various fields like modelling rates of change, describing physical phenomena, solving optimization problems, Music and Sound Engineering and in Biology and Medicine

1. Domain of rational functions

The domain of a rational function, $R(x) = \frac{f(x)}{g(x)}$ is the set of all real numbers except numbers for which $g(x)$ is zero.

Examples

1. What is the domain of the function $y = \frac{1}{x-2}$

Solution:

The domain of y is the set of all real numbers except $x = 2$. That is, domain of y is $\{x : x \in R, x \neq 2\}$

2. Determine the domain of the following rational functions;

   a. $f(x) = \frac{5x-3}{2x-14}$

   b. $p(x) = \frac{3x+1}{x^2-9}$

   c. $q(x) = \frac{3x+8}{2x-7}$

Solutions:

a. Given that $f(x) = \frac{5x-3}{2x-14}$, let the denominator $2x - 14 = 0$

$\Rightarrow 2x = 14$

$x = 7$

Thus, the domain of $f\left(x\right)$ is real except $x = 7$. That is $\{x : x \in R, x \neq 7\}$

b. Given $p(x) = \frac{3x+1}{x^2-9}$, let $x^2 - 9 = 0 \Rightarrow (x-3)(x+3) = 0 \Rightarrow x = 3$ *or* $x = -3$

Domain: $\{x : x \in R, x \neq 3 \text{ or } \neq -3\}$

c. Given $q(x) = \frac{3x+8}{2x-7}$, let $2x - 7 = 0 \Rightarrow 2x = 7 \Rightarrow x = \frac{7}{2}$

Domain: $\left\{x : x \in R, x \neq \frac{7}{2}\right\}$

2. Range of rational functions

The range of a rational function refers to the set of all possible output values (y-values) it can produce. That is the range is the same as the domain of the inverse function. Thus, in finding the range, we first find the inverse function, then its domain, which is the range.

Example

Determine the range of the following rational functions;

a. $f(x) = \frac{5x-3}{2x-14}$

b. $p(x) = \frac{3x+8}{2x-7}$

Solution:

a. Given $f(x) = \frac{5x-3}{2x-14}$

let $f(x) = y$

$\Rightarrow y = \frac{5x-3}{2x-14}$ making x the subject

$y(2x-14) = 5x-3$

$2xy - 14y = 5x - 3$

$2xy - 5x = 14y - 3$

$x(2y-5) = 14y - 3$

Thus, the inverse function is;

$x = \frac{14y-3}{2y-5}$

Finding the domain of the inverse function, let $2y - 5 = 0 \Longrightarrow y = \frac{5}{2}$

Range: $\left\{y : y \in R, x \neq \frac{5}{2}\right\}$

b. Given $p(x) = \frac{3x+8}{3x-7}$

Let $p(x) = y$

$\Rightarrow y = 3\frac{x+8}{3x-7}$

$y(2x-7) = 3x+8$

$2xy - 7y = 3x + 8$

$2xy - 3x = 7y + 8$

$x(2y-3) = 7y + 8$

$x = \frac{7y+8}{2y-3}$ which is the inverse function

For the domain of the inverse, let $2y - 3 = 0 \Longrightarrow y = \frac{3}{2}$

Range: $\left\{y : y \in R, x \neq \frac{3}{2}\right\}$

3. Zeroes of rational functions

The zeroes, or roots of rational functions, are the specific input values ($x - values$) that cause the entire function's output ($y - value$) to become zero. They basically "cancel out" the numerator when the denominator doesn't equal zero. Finding the zeroes of rational functions involves solving the equation formed by setting the function equal to zero and manipulating the expression to isolate x.

Examples

1. What is/are the zeros of the function $f : x \rightarrow \frac{x+3}{x-2}$

Solution:

The zeros are computed as $x + 3 = 0 \implies x = -3$

2. Find the zeros of the function defined by $f\colon x \to \frac{2x-1}{2x^2-9x-5}$

Solution:

To find the zeros of $f(x) = \frac{2x-1}{2x^2-9x-5} = 0 \implies 2x - 1 = 0 \implies x = \frac{1}{2}$

Thus, the zeros of $f(x) = \frac{1}{2}$

Theme or Focal Area(s): Basic operations on rational functions

Basic arithmetic operations involving rational functions may be performed.

Examples

Given that $f(x) = \frac{2x-1}{x-2}$ and $g(x) = \frac{x-3}{x+3}$. Simplify

a. $f(x) - g(x)$

b. $f(x) + g(x)$

Solutions:

a. $f(x) - g(x) = \frac{2x-1}{x-2} - \frac{x-3}{x+3}$

$$= \frac{(2x-1)(x+3) - (x-3)(x-2)}{(x-2)(x+3)}$$

$$= \frac{(2x^2 + 6x - x - 3) - (x^2 - 2x - 3x + 6)}{(x-2)(x+3)}$$

$$= \frac{(2x^2 + 5x - 3) - (x^2 - 5x + 6)}{(x-2)(x+3)}$$

$$= \frac{x^2 + 10x - 9}{(x-2)(x+3)}$$

b. $f(x) + g(x) = \frac{2x-1}{x-2} + \frac{x-3}{x+3}$

$$= \frac{(2x-1)(x+3) + (x-3)(x-2)}{(x-2)(x+3)}$$

$$= \frac{(2x^2 + 6x - x - 3) + (x^2 - 2x - 3x + 6)}{(x-2)(x+3)}$$

$$= \frac{(2x^2 + 5x - 3) + (x^2 - 5x + 6)}{(x-2)(x+3)}$$

$$= \frac{3x^2 + 3}{x - 2x + 3}$$

Theme or Focal Area(s): Decomposition into rational functions

Decomposition into partial fractions is a technique used to simplify and decompose a rational function into a sum of simpler fractions. It is particularly useful when dealing with complex rational functions, making integration, differentiation, which makes solving equations more manageable.

Equivalence of expressions

Example

Find the values of the constants A, B and C such that

$x^2 - 4x + 5 = A(x - 1)(x + 2) + B(x + 2)(x - 4) + C(x - 4)(x - 1)$

Solution:

$x^2 - 4x + 5 = A(x - 1)(x + 2) + B(x + 2)(x - 4) + C(x - 4)(x - 1)$

$x^2 - 4x + 5 = A\,x^2 + Ax - 2A + B\,x^2 - 2Bx - 8B + C\,x^2 - 5Cx + 4C$

$x^2 - 4x + 5 = A\,x^2 + B\,x^2 + C\,x^2 + Ax - 2Bx - 5Cx - 2A - 8B + 4C$

$x^2 - 4x + 5 = (A + B + C)\,x^2 + (A - 2B - 5C)x - 2A - 8B + 4C$

By comparison, $A + B + C = 1$,

$A - 2B - 5C = -4$ and

$-2A - 8B + 4C = 5$

Solving the three equations simultaneously,

$A = \frac{5}{18}$, $B = -\frac{2}{9}$ and $C = \frac{17}{18}$

Decomposition in Partial Fractions

Rational expressions with linear factors as denominators can be decomposed as follows:

$$\frac{(ax - b)}{((x - p)(x + q)(x - r)\ldots)} = \frac{A}{(x - p)} + \frac{B}{(x + q)} + \frac{C}{(x - r)}\ldots$$

Rational expressions with repeated factors as denominators can be decomposed as follows:

$$\frac{ax - b}{(x - p)^3} = \frac{A}{(x - p)} + \frac{B}{(x - p)^2} + \frac{C}{(x - p)^3}$$

Rational expressions with irreducible quadratic factors as denominators can be decomposed as follows:

$$\frac{ax - b}{a\,x^2 + bx + c} = \frac{Ax + B}{a\,x^2 + bx + c}$$

Rational expressions with numerators having degrees greater or equal to the degree of the denominator are deemed improper. Such expressions need to be reduced by long division before decomposed into partial fractions

Examples

Resolve $\frac{2x^2 + 1}{(x - 1)(x + 2)}$ in partial fractions.

Solution:

$$\frac{2x^2+1}{(x-1)(x+2)} = 2 + \frac{-2x+5}{(x-1)(x+2)}$$

$$= 2 + \frac{A}{x-1} + \frac{B}{x+2}$$

$$= \frac{2(x-1)(x+2)}{(x-1)(x+2)} + \frac{A}{x-1} + \frac{B}{x+2}$$

$$= \frac{2(x-1)(x+2) + A(x+2) + B(x-1)}{(x-1)(x+2)}$$

$2x^2+1 = 2(x-1)(x+2) + A(x+2) + B(x-1)$

$2x^2+1 = 2x^2 + 2x - 4 + Ax + 2A + Bx - B$

$2x^2+1 = 2x^2 + (A+B+2)x + 2A - B - 4$

By comparison, $A + B + 2 = 0$ and

$2A - B - 4 = 1$

Solving the equations simultaneously yields $A = 1$ and $B = -3$

$$\therefore \frac{2x^2+1}{(x-1)(x+2)} = 2 + \frac{1}{(x-1)} - \frac{-3}{(x+2)}$$

Learning Task

Guide learners to solve the following task to check for understanding. Support system should be offered to learners who will struggle.

Learners to

a. review the concept of rational numbers and functions.

b. develop and extend to concept of rational numbers.

c. recall concept of domain, range and zeros of functions and apply concept to rational functions.

d. solve basic arithmetic operations on rational functions.

e. split and simplify complex rational functions into sums of simpler partial fractions.

f. solve for values of constants of functions.

Pedagogical Exemplars

The objectives for this week's lessons includes helping Learners to recognise rational functions, find the domain, range and zeros of rational functions as well as perform basic arithmetic operations on them. Learners are also to be able to decompose complex rational functions into simpler partial fractions and find constants of functions. The following pedagogical approaches are proposed in the curriculum are to be taken into consideration:

a. Using talk-for-learning: recall key facts about polynomial functions. Attention should be given to learners to make correction of their output.

b. In collaborative and well-supervised groups: Learners examine and explore graphically and by algebraic means the set of values for which a given rational function will exist (domain of the function). Ensure that you attend to learners with low and medium learning abilities as well as high ability learners.

c. **In collaborative and well-supervised groups:** Learners examine and explore graphically and by algebraic means the values of x of a function for which the function is zero

d. **Using talk for learning in a small mixed ability/gender group,** Learners perform basic operations on rational functions.

e. **Talk for learning:** Learners will be working in convenient groups (ability, mixed ability, mixed gender, or pairs etc.) to explore factorization and expansion of denominators and numerators to simplify rational functions

f. **Talk for learning:** Learners will be working in convenient groups (ability, mixed ability, mixed gender, or pairs etc.) to apply knowledge of equivalence to determine values of constants of functions

g. **Talk for learning:** Learners will be working in convenient groups (ability, mixed ability, mixed gender, or pairs etc.) to split rational functions into partial fractions

h. **Talk for learning:** Learners will be working in convenient groups (ability, mixed ability, mixed gender, or pairs etc.) to determine values of constants of functions

Key Assessment

Assessment level 1: Recall:

1. What is the restriction on the domain of the rational function $f(x) = \frac{2x+1}{x-3}$?
2. The range of the function $g(x) = \frac{x^2}{x-1}$ excludes what value/s?
3. What are the zeros of the function $h(x) = (x + 2)(x - 5)$?
4. If $f\left(x\right) = \frac{x+1}{x-2}$ and $g\left(x\right) = \frac{2x-3}{x+1}$ what is the numerator of $\frac{f(x)}{g(x)}$?
5. What is the decomposed form of the function $j(x) = \frac{1}{x^2-4}$?

Assessment Level 2: Skills of conceptual understanding:

1. Why is the function $k\left(x\right) = \frac{x^2+1}{x-2}$ defined for all real numbers except $x = 2$.
2. Considering the function $f(x) = \frac{x+1}{x^2}$. How does the range change if we modify the function to $f'(x) = \frac{x+1}{x^2+1}$? Briefly explain.
3. The function $h(x) = \frac{x^2-4}{x+2}$ has a zero at $x = 2$. Does this necessarily mean the original function $h'(x) = x^2 - 4$ will also have a zero at $x = 2$? Briefly explain.
4. Given $f\left(x\right) = \frac{2x+1}{x}$ and $g\left(x\right) = \frac{x-3}{x+2}$, find the product $f(x) \times g(x)$.
5. Given that $f(x) = \frac{2x+3}{x^2-9}$, find the constants A and B in the following form: $\frac{A}{x+3} + \frac{B}{x-3}$

Assessment Level 3: Strategic reasoning:

1. Resolve each of the following into partial fractions

   a. $\frac{3x^3 + 5x^2 - 7x + 3}{(x^2-1)^2}$

   b. $\frac{5x^3 + 13x - 1}{(x^2+4)^2}$

2. A rectangular storage container with a square base and open top is being constructed. The material for the base costs ¢2 per square foot, and the material for the side's costs ¢1 per linear foot. Let x represent the side length (in feet) of the base.

3. Express the total material cost (C) as a rational function of x. (Hint: Consider areas of different parts)
4. Find the rate of change of the total material cost with respect to the side length (i.e., C'(x)). Interpret the meaning of C'(x) in the context of this problem.

Assessment Level 4: Extended critical thinking and reasoning:

1. A chemical reaction in a manufacturing process is described by the following rational function:
2. $R(t) = \frac{3t+2}{t^2+5t+6}$
3. where *R(t)* represents the rate of the chemical reaction at time *t*. Perform partial fraction decomposition to analyse the behaviour of the reaction over time.
4. Find the values of the constants A, B and C such that
   $x^2 - 4x + 5 = A(x-1)(x+2) + B(x+2)(x-4) + C(x-4)(x-1)$
5. Graph the rational function $p(x) = \frac{3x^2 - 2x - 1}{x^2 + x - 2}$
6. A company manufactures and sells a particular electronic device. The demand (D) for the device is modelled by the rational function $D(p) = \frac{10000}{p+2}$, where p represents the price (in cedis) of the device. The total cost (C) of producing x devices is modeled by the rational function $C(x) = \frac{3x^2 + 12x + 50}{x}$.
   a. Express the profit (P) as a function of the price (p) considering both demand and production cost (P(p) = Revenue - Cost).
   b. Assuming the company wants to set the price to maximize their profit, what price (p) would they choose? How many devices (x) would they produce and sell at that price to achieve maximum profit? (Hint: This requires analysing the critical points of $P(p)$ and relating them to the context of the problem).

Section 3 Review

This section, which covered weeks six (6) to eleven (11), focused on recognising, understanding and applying algebraic concepts to solve real-world problems. We explored various sub-topics within the theme of Modelling with Algebra. The learner is required to have gone through and understood to their various ability levels the following weekly focal areas.

Week 6: Sequences

a. We delved into the concept of sequences, ordered lists of numbers with a specific pattern.
b. We learned to identify number patterns, determine rules for finding terms, and calculate the nth term (any term in the sequence).
c. Different types of sequences (arithmetic, geometric, etc.) were explored, along with practical applications in areas like finance (compound interest) or physics (falling objects).

Week 7: Relations and Functions

a. We shifted our focus to relations, connections between sets of numbers.
b. We distinguished between relations and functions (special relations with a unique output for each input).
c. Evaluating functions (finding the output for a specific input) became a key skill.
d. We explored various types of functions (linear, quadratic, etc.) and identified their attributes (domain, range, etc.).

e. The concept of composite functions, where the output of one function becomes the input for another, was introduced.

Week 8 & 9: Graphs and Applications

a. We focused on visualizing functions through graphs – lines for linear functions and curves for quadratics.

b. Identifying key features of these graphs, like intercepts (x and y-axis intersections), became crucial.

c. Week 9 delved into applications of linear functions, including calculating areas enclosed by graphs and solving systems of linear equations (representing two or more equations working together) with real-world word problems.

Week 10: Quadratic Functions and Applications

a. Week 10 built upon our understanding of quadratic functions, focusing on finding their roots (solutions to the equation when the function equals zero).

b. We explored factoring techniques to manipulate quadratic expressions and the Factor and Remainder Theorems, which connect factors of a polynomial function to its roots and remainders.

c. Week 10 concluded by examining how to graph these functions and identify their applications in real-life scenarios.

Week 11: Rational Functions

a. The final week introduced rational functions, involving expressions with variables in both the numerator and denominator.

b. We learned about their domain (all possible input values) and zeroes (inputs that make the function zero), which are crucial for understanding their behaviour.

c. We were introduced to the concept of decomposing rational functions into partial fractions, a technique used to simplify their representation for further analysis.

This section has equipped you with a powerful toolkit for applying algebraic concepts to model and solve problems involving sequences, functions, and various types of graphs. You can now confidently tackle real-world scenarios by translating them into mathematical models and using your algebraic skills to find solutions.

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SECTION 4: MATRICES

Strand: **Modelling with Algebra**

Sub-Strand: Number and Algebraic Patterns

Content Standard: Demonstrate knowledge and understanding of algebraic processes and reasoning in relation to matrix operations

Learning Outcome (s):

1. *Identify and describe the order of a matrix, the identity matrix and Zero matrix*
2. *Find the determinant of 2 × 2 matrices*
3. *Perform basic arithmetic operations on 2 by 2 matrices (addition, subtraction and multiplication)*

INTRODUCTION AND SECTION SUMMARY

A matrix is a rectangular arrangement of numbers, symbols, or expressions enclosed within brackets ( ) *or* [ ]. Each element within the matrix has a specific location identified by its row (horizontal position) and column (vertical position). We describe a matrix by its dimensions, specifying the number of rows and columns. For example, a matrix with 2 rows and 3 columns are called a 2 × 3 matrix. When a matrix has the same number of rows and columns (for example, 3 × 3), it is called a **square matrix.** A square matrix with non-zero entries only on its main diagonal (from top left to bottom right) is called a **diagonal matrix.** A special square matrix with ones (1s) on its main diagonal and zeros (0s) elsewhere is called an identity matrix. It plays a crucial role in solving matrix equations. Matrices have extensive applications in various fields: They are used to organize and analyse large datasets in statistics, health, economics, and social sciences. In computer graphics, matrices are essential for representing 3D objects, transformations, and lighting effects in computer graphics. Matrices are also used to analyse electrical circuits and solve complex problems related to currents and voltages. They are applied in physics to represent physical systems like forces and motion, simplifying calculations and analysis.

The week covered by the section is Week 12:

- **a.** Definition, order and types of matrices
- **b.** Finding the determinants of 2 × 2 matrices
- **c.** Arithmetic operations of matrices
- **d.** Multiplication of matrices

SUMMARY OF PEDAGOGICAL EXEMPLARS

This section requires hands-on activities where learners engage in practical activities on matrices and basic operations on matrices. Learners should be given the platform to work in groups to develop their own real-life questions and find answers. Therefore, Experiential learning activities and Mixed-ability groupings should dominate the lessons on these concepts. All learners, irrespective of their learning abilities should be assisted to take part fully in investigations and presentation of findings. However, make considerations and accommodations for the different groups. That is, offer approaching proficiency learners the opportunity to make oral presentations. Then, extend activities for the above average/highly proficient learners to use formulae and computer applications to solve problems.

ASSESSMENT SUMMARY

Assessment methods ranging from quizzes, tests, and homework assignments can be used to evaluate learners understanding of concepts and their ability to solve problems. Performance tasks like solving real-world problems involving matrices will also be used to assess learner's application of these mathematical skills. Also, make use of various visual aids and charts on graphs as well as local games like Oware as teaching and learning material (TLMs) will also be incorporated to engage learners in hands-on learning experiences. Assessment strategies which vary from Level 1 to Level 4 questions of the DoK will be used. Teachers should record the performances of learners for continuous assessment records.

Week 12

Learning Indicator(s):

1. *Recognise a matrix including types of matrices and state its order*
2. *Find the determinant of a 2x2 matrix*
3. *Add and subtract matrices (2x2 matrix)*
4. *Multiply a matrix by a scalar and a matrix by a matrix (2x2 matrices)*

Theme or Focal Area: Definition, order and types of matrices

A matrix (plural: matrices) is a rectangular array of numbers, symbols, or expressions, organized in rows and columns. Each entry in a matrix is called an element or an entry, and it is identified by its row and column indices.

$$\begin{array}{c c} & 4\ columns \\ 2\ rows \begin{array}{c}\rightarrow \\ \rightarrow\end{array} & \begin{bmatrix} \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 7 & 3 & 6 \\ -2 & 1 & 3 & 5 \end{bmatrix} \end{array}$$

Everyday situations that exemplify the concept of matrices include

a. Classroom sitting arrangement
b. Provision items in a shop
c. A pack of bottled water, among others.

Example:

Suppose that we wish to express the information of possession of pens and pencils by Afiba and his two friends Enyonam and Nana, which is as follows:

Afiba has 2 pens and 7 pencils,

Enyonam has 1 pen and 5 pencils,

Nana has 3 pens and 2 pencils.

Now, this could be arranged in tabular form as;

Individuals	Items	
	Pens	**Pencils**
Afiba	2	7
Enyonam	1	5
Nana	3	2

Which could be expressed in matrix form as;

$$A = \begin{pmatrix} 2 & 7 \\ 1 & 5 \\ 3 & 2 \end{pmatrix} \text{ or } A = \begin{bmatrix} 2 & 7 \\ 1 & 5 \\ 3 & 2 \end{bmatrix}$$

Order or dimension of a matrix

An $m \times n$ matrix has m rows (horizontal) and n columns (vertical). Each element of a matrix is denoted by a variable with subscripts. For example, $\boldsymbol{a}_{23,}$ represents the element in the 2nd (second) row and 3rd (third) column of the matrix.

$$\text{Example, the matrix } A = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ \vdots \\ m \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & \dots & n \end{matrix} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{matrix}$$

is an $m \times n$ dimensional matrix having m number of rows and n number of columns.

The order of the matrix $M = \begin{pmatrix} -3 & 4 & -1 \\ 5 & 2 & 0 \end{pmatrix}$ is 2×3

Equality of Matrix

Two matrices, A and B are said to be equal if they have the same order and their corresponding elements are equivalent.

For example, $M = \begin{pmatrix} 3 & 5 & -2 & \frac{3}{4} \\ 4 & \frac{1}{2} & 0 & -\frac{2}{5} \end{pmatrix}$ and N= $\begin{pmatrix} 3 & 5 & -2 & \frac{3}{4} \\ 4 & \frac{1}{2} & 0 & -\frac{2}{5} \end{pmatrix}$ are two equal matrices

since they have the same dimension and have their corresponding elements the same.

Example

If $P = \begin{pmatrix} \frac{3}{2} & x \\ y - x & -3 \end{pmatrix}$ and $Q = \begin{pmatrix} x & 1.5 \\ 3.5 & -3 \end{pmatrix}$ are two equal matrices, find the values of x and y.

Solution:

Since $P = Q$,

$\frac{3}{2} = x$ and $y - x = 3.5$

$y - \frac{3}{2} = 3.5$

$y = 5$

Zero matrix

A zero matrix is a matrix whose entries are all zeros or are equivalent to zero $(0 \quad 0)$, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -i + i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x - x \end{pmatrix}$ are all zero matrices

Example:

Which of the following matrices is/are a zero matrix (matrices)?

a. $A = \begin{pmatrix} x - x & 0 & 0 \\ 0 & -b + b & 0 \end{pmatrix}$

b. $B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

c. $C = \begin{pmatrix} m - m & 0 \\ 0 & 0 \end{pmatrix}$

Solution:

a. Simplifying the entries in matrix A gives $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, thus A is a zero matrix

b. Matrix B has two (2) non-zero entries, thus B is not a zero matrix

c. Matrix C can be simplified as $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ making it a zero matrix

Theme or Focal Area: Determinants of 2 × 2 matrices

Imagine a woven basket from village, but instead of holding your favourite fruits, it holds numbers arranged in a neat grid, like two rows of cowrie shells. This grid is what mathematicians call a 2x2 matrix. In Ghana, we have a word for unlocking secrets – "Odomankoma" (key). Determinants act as the Odomankoma for these matrices, revealing a unique property based on how the numbers are arranged.

Here's a breakdown for 2 × 2 matrices, like our basket of numbers:

Basic Structure: A 2 × 2 matrix looks like this:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (where a represents the element in the first row, first column; b represents the element in the first row, second column, and so on)

Determinant Formula: There's a special formula to calculate the (determinant, *det*) of a 2 × 2 matrix: $det = (a \times d) - (b \times c)$

where a, b, c and d represent the elements (oduas or twigs) of the matrix as shown above.

Visualizing the Determinant: Imagine drawing a diagonal line across the basket, starting from the top left corner (a) and reaching the bottom right corner (d). Now, draw another diagonal line starting from the top right corner (b) and reaching the bottom left corner (c). The determinant captures the difference between the product of the elements along one diagonal ($a \times d$) and the product of elements along the other diagonal ($b \times c$).

Example:

Consider a 2 × 2 matrix:

$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

Using the formula, the determinant (*det*) would be:

$\det(A) = (2 \times 4) - (3 \times 1) = 8 - 3 = 5$

Importance:

The determinant of a 2 × 2 matrix has various applications, including:

a. **Solving systems of linear equations**: Determinants play a crucial role in finding solutions to systems of linear equations with two variables.

b. **Invertibility of matrices:** A non-zero determinant indicates that the matrix is invertible, meaning it has an inverse matrix.

c. **Area calculation:** In specific contexts, the determinant can be used to calculate the area enclosed by a parallelogram defined by the matrix's row vectors. Imagine a farmer needs to calculate the area of a rectangular plot of land represented by a 2 × 2 matrix, where each element represents the length and width of the plot in meters. The determinant, in this case, can be used to calculate the area (**note**, this application has limitations for general area calculation)

Example

1. Evaluate the determinants of the following matrices

a. $\begin{vmatrix} 21 & 4 \\ 17 & 9 \end{vmatrix}$

b. $\begin{vmatrix} 2 & -8 \\ -3 & 6 \end{vmatrix}$

c. $\begin{vmatrix} a+3 & 7-a \\ a & 7 \end{vmatrix}$

Solutions:

Given a 2 × 2 matrix $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, determinant of A, $det(A) = ad - bc$

a. $det\begin{vmatrix} 21 & 4 \\ 17 & 9 \end{vmatrix} = (21 \times 9) - (4 \times 17) = 121$

b. $\det\begin{vmatrix} 2 & -8 \\ -3 & 6 \end{vmatrix} = ((2 \times 6) - (-8 \times -3)) = -12$

c. $\det\begin{vmatrix} a+3 & 7-a \\ a & 7 \end{vmatrix} = (7(a+3) - a(7-a)) = 7a + 21 - 7a + a^2 = a^2 + 21$

Theme or Focal Area: Arithmetic operations of matrices

Addition of matrices

Two matrices, A and B can only be added if and only if they have the same dimension

Given that $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$,

$$A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

Example

If $A = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$, find

a) $A + B$

b) $B + A$

c) $A + (B + C)$

d) $(A + B) + C$

e) What is the relationship between your answers in a) and b)

f) What is the relationship between your answers in c) and d)

Solution:

a) $A + B = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}$

$= \begin{pmatrix} -5+4 & 3-3 \\ 2+7 & -1-5 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 9 & -6 \end{pmatrix}$

b) $B + A = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} + \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 4-5 & -3+3 \\ 7+2 & -5-1 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 9 & -6 \end{pmatrix}$

c) $B + C = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$

$= \begin{pmatrix} 4-1 & -3+3 \\ 7+5 & -5+4 \end{pmatrix}$

$= \begin{pmatrix} 3 & 0 \\ 12 & -1 \end{pmatrix}$

$A + (B + C) = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 12 & -1 \end{pmatrix}$

$= \begin{pmatrix} -5+3 & 3+0 \\ 2+12 & -1-1 \end{pmatrix}$

$= \begin{pmatrix} -2 & 3 \\ 14 & -2 \end{pmatrix}$

d) $(A + B) + C = \begin{pmatrix} -1 & 0 \\ 9 & -6 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$

$= \begin{pmatrix} -1-1 & 0+3 \\ 9+5 & -6+4 \end{pmatrix}$

$= \begin{pmatrix} -2 & 3 \\ 14 & -2 \end{pmatrix}$

e) $A + B = B + A$ therefore, matrix addition is commutative

f) $A + (B + C) = (A + B) + C$ therefore, matrix addition is associative

Subtraction of matrices

Two matrices, A and B can only subtracted if and only if they have the same dimension

$$A - B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{pmatrix}$$

Example

If $A = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$, Evaluate the following

a) $A - B$

b) $B - A$

c) $A - (B - C)$

d) $(A - B) - C$

e) What is the relationship between your answers in a) and b)

f) What is the relationship between your answers in c) and d)

Solution:

a) $$A - B = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}$$
$$= \begin{pmatrix} -5 - 4 & 3 - (-3) \\ 2 - 7 & -1 - (-5) \end{pmatrix}$$
$$= \begin{pmatrix} -9 & 6 \\ -5 & 4 \end{pmatrix}$$

b) $$B - A = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} + \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 4 - (-5) & -3 - 3 \\ 7 - 2 & -5 - (-1) \end{pmatrix}$$
$$= \begin{pmatrix} 9 & -6 \\ 5 & -4 \end{pmatrix}$$

c) $$B - C = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} - \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 4 - (-1) & -3 - 3 \\ 7 - 5 & -5 - 4 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & -6 \\ 2 & -9 \end{pmatrix}$$

$$A - (B - C) = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} 5 & -6 \\ 2 & -9 \end{pmatrix}$$
$$= \begin{pmatrix} -5 + 5 & 3 - (-6) \\ 2 - 2 & -1 - (-9) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 9 \\ 0 & 8 \end{pmatrix}$$

d) $$(A - B) - C = \begin{pmatrix} -9 & 6 \\ -5 & 4 \end{pmatrix} - \begin{pmatrix} -1 & 3 \\ 5 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} -9 - (-1) & 6 - 3 \\ -5 - 5 & 4 - 4 \end{pmatrix}$$
$$= \begin{pmatrix} -8 & 3 \\ 0 & 0 \end{pmatrix}$$

e) $A - B \neq B - A$ therefore, matrix subtraction is not commutative

f) $A - (B - C) \neq (A - B) - C$ therefore, matrix subtraction is not associative

Theme or Focal Area: Multiplication of matrices

Scalar multiplication of matrices

Given a matrix, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a scalar, k

$$kM = k\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} ak & bk \\ ck & dk \end{pmatrix}$$

Example

Given that $A = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ and the matrix $P = -2A$, write out the matrix P

Solution:

$P = -2A$

$$= -2\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ -6 \\ -4 \end{pmatrix}$$

Multiplication of matrices

Matrix multiplication, unlike multiplying individual numbers, involves a specific process for combining elements from two matrices to create a new matrix. The two matrices must have compatible dimensions for multiplication. The number of columns in the first matrix $A_{m\times n}$ must equal the number of rows in the second matrix $B_{p\times q}$. The resulting product matrix will have dimensions $m \times q$. We don't directly multiply corresponding elements between the matrices. Instead, to find an element at any row (i) and column (j) of the resulting product matrix, we take the dot product of row i from the first matrix (A) with column j from the second matrix (B). The dot product involves multiplying corresponding elements between the row and column vectors and summing those products. We repeat this process for each element in the resulting product matrix, considering all possible row-column combinations.

Example given a matrix, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$,

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} \text{ Multiplying } N \text{ by the } 1st \text{ row of } M$$

Which give the entries for first row elements of the product as $(ae + bg \quad af + bh)$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} \text{ Multiplying } N \text{ by the } 2nd \text{ row of } M$$

Giving the entries for first row elements of the product MN as $(ce + dg \quad cf + dh)$

The final result will be; $MN = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$

NOTE: Matrix multiplication does not commute $(AB \neq BA)$ in general. The order of the matrices matters when multiplying them.

Examples

Given that $A = \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 3 \\ -2 & -7 \end{pmatrix}$

Evaluate the following;

a) AB

b) BA

c) what is the relationship between your answers in a) and b)

Solution:

a. $AB = \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ -2 & -7 \end{pmatrix}$

$$= \begin{pmatrix} 4(-2) + (-2)(-2) & 4(3) + (-2)(-7) \\ 1(-2) + 3(-2) & 1(3) + 3(-7) \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 26 \\ -8 & 19 \end{pmatrix}$$

b. $BA = \begin{pmatrix} -2 & 3 \\ -2 & -7 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} -2(4) + 3(1) & (-2)(-2) + 3(3) \\ (-2)(4) + (-7)(1) & (-2)(-2) + (-7)(3) \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 13 \\ -15 & -17 \end{pmatrix}$$

c) $AB \neq BA$ therefore, matrix multiplication is not commutative

Learning Tasks

Guide learners to perform the following task to check for understanding. Support system should be offered to learners who will struggle.

Learners to

a. review and define matrices, matrix order and types.

b. find the determinants of 2×2 matrices

c. perform basic arithmetic operations of matrices

d. multiply two or more 2 × 2 of matrices

Pedagogical Exemplars

The objectives for this week's lessons includes helping learners to be able demonstrate knowledge and understanding of algebraic processes and reasoning in relation to matrix operations. The following pedagogical strategies are recommended in the curriculum:

Collaborative Learning: Learners will be working in convenient groups (ability, mixed ability, mixed gender, or pairs etc.) to:

a. recognize the condition necessary for matrix addition and subtraction

b. investigate matrix addition and subtraction

c. explore commutative and associative properties of addition

Experiential Learning: Learners engage in hands-on activity (learning by doing) to

a. create matrices

b. define matrices

c. determine the order of matrices.

Enquiry Based Learning: Learners use research resources (textbooks, electronic devices and any additional relevant resources) to discover matrices in real life

Key Assessment

Assessment Level 1: Recall

a. Given that $\begin{bmatrix} 3y+1 & 7 \\ 2y+x & 3 \end{bmatrix} = \begin{bmatrix} 10 & 3y-x \\ 8 & 3 \end{bmatrix}$ find the values of x and y

b. Consider the following matrices:

$$A = \begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix}, B = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, C = \begin{bmatrix} -2 & 1 & 3 \\ 4 & -5 & 2 \end{bmatrix}, \textit{and } D = \begin{bmatrix} -2 & 1 \\ 5 & 3 \end{bmatrix}$$

Complete each calculation below, if not possible, explain why not

a. A+D

b. $A \cdot B$

c. $D \cdot C$

d. $6 \cdot C$

c. If $A = \begin{pmatrix} 3 & 5 \\ -7 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & -2 \\ 6 & 1 \end{pmatrix}$

Find

i. $A + B$

ii. $B - A$

d. Consider two matrices:

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 1 \\ 2 & 0 \end{pmatrix}$

i. Find the sum (A + B).

ii. Find the difference (A - B).

e. Consider two matrices: $C = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ Multiply matrix C by matrix D

f. Given that $A = \begin{bmatrix} 2m+1 & -1 & mn+4 \\ 8 & -m+7 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 5-3n & -2 \\ n-2m & 10 & -n+\frac{2}{3}m \end{bmatrix}$ and $A = B$, find the values of m and n

Assessment Level 2: Skills of Conceptual Understanding

a. Explain why the determinant formula only works for 2x2 matrices. How would the concept change for larger matrices?

b. Can a matrix have a non-zero determinant even if all its elements are zero? Explain your answer.

c. Explain the difference between the determinant and a single element within a 2×2 matrix.

d. How does the determinant change if we swap the positions of two rows (or columns) in a 2×2 matrix?

e. What is the determinant of the identity matrix.

g. Given that $X = \begin{pmatrix} 4 & 0 \\ 2 & -1 \end{pmatrix}$ *and* $Y = \begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}$, find

i. XY

ii. YX

iii. What do your answers in i) and ii) show you about multiplication of matrix?

Assessment Level 3: Strategic Thinking

a. Given a 2×2 matrix with determinant 0, what can you infer about the relationship between the rows (or columns) of the matrix?

b. Two 2×2 matrices, A and B, have the same determinant. If we multiply matrix A by a constant k, how does the determinant of the resulting matrix (kA) compare to the determinant of B?

c. Is it possible to find the determinant of a 2×2 matrix without memorizing the formula? Explain your answer.

Section 4 Review

In this section, we delved into matrices which are rectangular arrays of numbers used to represent and manipulate data. We explored various concepts within the theme of Modelling with Algebra, equipping you with tools to tackle problems involving these powerful mathematical objects.

We began by defining matrices, specifying their order (dimensions like 2×2 *or* 3×4 based on the number of rows and columns) and exploring different types (square matrices with equal rows and columns, diagonal matrices with non-zero entries only on the diagonal, etc.).

A key concept introduced was the determinant, a special number calculated from a matrix that captures a specific property of its elements' arrangement. We learned techniques for finding determinants for 2x2 matrices.

We explored various arithmetic operations applicable to matrices of the same order. These included:

Addition: Adding corresponding elements from matrices of the same size.

Subtraction: Subtracting corresponding elements from matrices of the same size.

Scalar multiplication: Multiplying each element of a matrix by a constant (scalar).

Matrix multiplication, a unique operation combining elements from two matrices to produce a new resulting matrix. We learned the rules for multiplying matrices, considering factors like matrix dimensions and order.

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